Online Appendix:
Gradualism and Rough Transition with Reference Dependence

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1 Microfoundation

In this section, I provide two microfoundations of the intrinsic payoff function π_i by showing the payoff functions in these two economic problems satisfy Assumptions 1 and 2.

1.1 Collusion in Cournot Competition

The first one is collusion in Cournot competition. N firms with a common marginal cost of c face the inverse demand function $a - \beta(\sum x_i)$ where x_i is quantity supplied by firm i, $(a-c)/2N\beta \leq x_i \leq (a-c)/N\beta$, and $a > c$. The lower bound of x_i is set such that the aggregate supply achieves the monopolist's profit. The upper bound is set such that the price becomes non-negative. The profit for firm i is

$$\pi_i = \left(a - \beta \sum_{j \neq i} y_j - \beta x_i \right) x_i - c x_i.$$

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Let $\alpha_i = (a-c)/N\beta - x_i$. Then, $\bar{\alpha} = (a-c)/2N\beta$ and

$$\frac{d\tilde{\pi}(\alpha)}{d\alpha} = -\frac{d\left[(a-\beta Nx)\,x - cx\right]}{dx} = c - a + 2\beta Nx \begin{cases} > 0 & \alpha < \bar{\alpha}, \\ = 0 & \alpha = \bar{\alpha}, \end{cases}$$

Thus, Assumption 1 is satisfied. The best-response payoff is calculated as

$$b(\pi) = \frac{1}{4\beta}\left[(a-c)\cdot\frac{(N+1)-(N-1)\sqrt{1-4N\beta\pi/(a-c)^2}}{2N}\right]^2.$$

This b is convex, and $\pi = (a-c)^2/(N+1)^2\beta$ is the unique solution to $b(\pi) = \pi$, satisfying Assumptions 1 and 2.

1.2 Trade Liberalization

The second microfoundation is trade liberalization. I consider a partial equilibrium model of trade between two symmetric countries, home and foreign. This is a special case of the model of ?, which can accommodate asymmetric country sizes. I denote foreign variables by those with the superscript $*$. There are two goods, 1 and 2. The demand function for good $k \in \{1,2\}$ is $D_k(p_k) = A - Bp_k$ in both countries, where p_k is the price of good k. The supply function for good k is given by $X_k(p_k) = a_k + \beta p_k$ and $X_k^*(p_k^*) = a_k^* + \beta^* p_k^*$ where $\beta = \beta^*$. By assuming $a_1 - a_1^* = a_2^* - a_2 > 0$ and $a_1 = a_2^*$, home exports good 1 and imports good 2, while foreign exports good 2 and imports good 1. Each country imposes the specific tariff $t \in [0,T]$ on its import where $T = (a_2^* - a_2)/(B+\beta)$ is the lowest prohibitive tariff in this model. Consequently, $p_1^* = p_1 + t^*$ and $p_2 = p_2^* + t$. The welfare is the sum of consumer surplus, producer surplus, and tariff revenue, given by

$$W(t,t^*) = \sum_{k=1,2}\int_{p_k}^{A/B} D_k(u)du + \sum_{k=1,2}\int_{-a/\beta}^{p_k} X_k(u)du + t(D_m(p_m) - X_m(p_m)),$$

where m is 2 for home and 1 for foreign. Taking derivatives with respect to t and t^* yield

$$\begin{aligned}
\frac{\partial W(t,t^*)}{\partial t} &= M\left(1 - \frac{\partial p_2(t,t^*)}{\partial t}\right) + t\frac{\partial M}{\partial p_2}\frac{\partial p_2(t,t^*)}{\partial t} = \frac{a_2^* - a_2}{4} - \frac{3(B+\beta)}{4}t,\\
\frac{\partial W(t,t^*)}{\partial t^*} &= -M\frac{\partial p_1(t,t^*)}{\partial t^*} = -\frac{a_2^* - a_2}{4} + \frac{B+\beta}{4}t^*,
\end{aligned}$$

where $M = (a_2^* - a_2)/2 - (B+\beta)t/2$ is the net import of home, and the second equality in each row follows from the equilibrium good price. These derivatives imply the welfare equaling

$$W(t,t^*) = \frac{a_2^* - a_2}{4}(t - t^*) - \frac{B+\beta}{8}(3t^2 - t^{*2}) + C,$$

where C is constant. Let $\alpha = T - t$, $\bar{\alpha} = T$, and $\tilde{\pi}(\alpha) = W(T-\alpha, T-\alpha)$. Then, Assumption 1 is satisfied as:

$$\frac{d\tilde{\pi}(\alpha)}{d\alpha} = \frac{B+\beta}{2}t\begin{cases} = 0 & \alpha = \bar{\alpha} \\ > 0 & \alpha < \bar{\alpha} \end{cases}$$

The best-response payoff is calculated as:

$$b(\pi) = \frac{(a_2^* - a_2)^2}{24(B+\beta)} - \frac{a_2^* - a_2}{2}\sqrt{\frac{C-\pi}{B+\beta}} + \frac{3C-\pi}{2}$$

This function b is convex, and $\pi = -(a_2^* - a_2)^2/36(B+\beta) + C$ is the unique solution to $b(\pi) = \pi$, satisfying Assumptions 1 and 2.

2 Cooperation after a Transitory Shock

This appendix analyzes the response of cooperation to unexpected one-off payoff shocks. I begin with players who are enjoying high cooperation at $\bar{z}$ in the optimal

SPE. Suppose, after they make the cooperation decisions in period t ($\pi_t = g(\bar{z}) = \bar{z}$), that the players receive an one-off payoff shock ϵ such that $\underline{\pi} < \pi_t = \bar{z} + \epsilon < \bar{\pi}$. While $\bar{z}$ stays the same in the following periods, the reference point moves, reflecting their higher payoff experience.

A negative shock reduces the cooperation levels in subsequent periods. After the unexpected shock, the players must re-optimize their cooperation path. If the shock is negative ($\epsilon < 0$), the reference point in period $t+1$ becomes lower than $\bar{z}$: $r_{t+1} = \rho r_t + (1-\rho)[g^*(r_t) + \epsilon] < \bar{z}$. Consequently, Lemma 6 implies a lower cooperation level in period $t+1$ and a gradual return to $\bar{z}$ thereafter. The experience of a bad day makes them tolerant of potential losses in the deviation path, preventing an instant return.

Remarkably, a positive shock can also hinder cooperation in the following periods by making the reference point in period $t+1$ higher than $\bar{z}$. For example, suppose the discount factor and the persistence of a reference point are sufficiently low, like in Example 2. The shock can cause a significant drop in cooperation in period $t+1$ because of the steep downward slope of g^* in Figure 3.

Notably, cooperation with a short history is robust to a positive transitory shock. Instead of the players at $\bar{z}$, consider players that start cooperating with an initial reference point significantly lower than $\bar{z}$, facing a positive shock in an early period. Then, the reference point r_t has not risen much yet, maintaining the post-shock reference point r_{t+1} lower than $\bar{z}$. This r_{t+1} implies a cooperation level higher than that without the shock: $g^*(\rho r_t + (1-\rho)\left[g^*(r_t) + \epsilon\right]) > g^*(\rho r_t + (1-\rho) g^*(r_t))$.

3 Cooperation Path with Misbeliefs about Reference Points

This section analyzes the game in which players incorrectly believe that their reference points are permanently constant ($\rho = 1$), whereas the actual reference points evolve according to the same law of motion as the main text ($\rho < 1$). The players learn their current reference points (r_{t+1}) every period, realizing their misbeliefs in the previous period ($r_{t+1} \neq r_t$). Nevertheless, they keep assuming constant reference points ($r_{t+k+1} = r_{t+1}$ for all $k \in \mathbb{N}$). The path that those players in period 1 perceive to be "optimal" is the solution to the following problem.[1]

$$\begin{aligned} v^*(r_1) = \max_{\{\pi_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \left[\pi_t - \mathbb{1}\{r_t > \pi_t\} \cdot \eta(r_t - \pi_t)\right], \\ s.t. \forall t \in \mathbb{N}, \pi_t \in \Gamma(r_t, v^*), \\ \forall t \in \mathbb{N}, r_{t+1} = r_t, \end{aligned} \tag{1}$$

where the feasible set $\Gamma(r_t, v^*)$ is defined as

$$\Gamma(r_t, v^*) = \Big\{\pi_t \in [\underline{\pi}, \bar{\pi}] : b(\pi_t) - \mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta\left[r_t - b(\pi_t)\right] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta}\right]\underline{\pi} - \frac{\delta\eta}{1-\delta} r_t \leq v^*(r_t)\Big\}.$$

Following this "optimal" path, the players implement π_1. In the next period, the players realize r_2 is different from the value assumed before. Thus, they re-optimize the path to maximize the present value at that point, solving the same problem (1) with $r_2 = \rho r_1 + (1-\rho)\pi_1$ instead of r_1. One can interpret it as renegotiation taking place every period. I characterize $\{\pi_t\}_{t=1}^{\infty}$ produced by this process; this is a sequence of realized values, not the optimized path the players formulate in period 1.

The problem (1) assumes constant r_t, eliminating the dynamic effect from the current cooperation level. Thus, setting $\pi_t = \max \Gamma(r_t, v^*)$ every period is the solution, and the misbelief implies a constant cooperation path ($\pi_t = \pi_1$ and $v^*(r_t) = \pi_t/(1-\delta)$ for all t). I consider the case in which the first-best cooperation is not a steady state. Then, a steady state z' solves

$$b(z') - z' = \frac{\delta}{1-\delta}\left[z' - \underline{\pi}\right] + \frac{\delta\eta}{1-\delta}\left[z' - \underline{\pi}\right] \tag{2}$$

[1] I focus on those with Nash-reversion strategies.

The convexity of b implies two solutions to this equation: $\underline{\pi}$ and $\bar{z}' > \underline{\pi}$, implying $\bar{z}'$ is a unique steady state.[2]

Comparing eq. (2) with eq. (7) shows two effects that make the steady state $\bar{z}'$ differ from $\bar{z}$. First, players with misbelief do not realize that the best deviation raises their reference points; thus, they underestimate the future losses after deviation. Second, they do not expect their reference points to decline over time during the penalty phase, overestimating future losses. These two opposing effects make $\bar{z}'$ differ from $\bar{z}$, and the net effect is ambiguous. I assume that $\bar{z}'$ is sufficiently close to $\bar{\pi}$, similar to Assumption 5, ensuring $b(\pi_t) > r_t$ when $r_t > \bar{z}'$.[3] I consider $[\underline{\pi}, \bar{z}']$ and $(\bar{z}', \bar{\pi}]$ separately.

When the initial reference points are low, the players choose a level of cooperation above them, unknowingly raising the reference points. Given $r_t \in [\underline{\pi}, \bar{z}']$, π_t satisfies

$$b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta}\right]\underline{\pi} - \frac{\delta\eta}{1-\delta}r_t = \frac{\pi_t}{1-\delta}. \tag{3}$$

Eq. (2) and (3) imply $\pi_t > r_t$ for $r_t < \bar{z}'$. Furthermore, π_t increases with r_t. Thus, after realizing higher reference points in the next period, the players re-optimize their cooperation, choosing a higher level than before. The repetition of this process makes the level of cooperation converge to the steady state $\bar{z}'$. Thus, the cooperation path exhibits "gradualism" similarly to the model with correct beliefs in the main text.

The misbelief generates a force that decelerate the convergence pace from $r_1 < \bar{z}'$. The players eventually reach the steady state, but they never expect to do so. Thus, they underestimate the value of future cooperation, becoming more incentivized to deviate. This effect hinders their cooperation in every period, thereby slowing the convergence.[4]

[2] For the explanation of the existence of the second solution $\bar{z}'$, please see the proof of Lemma 5 in Appendix D, which rigorously proves the existence of the second solution for the model with correct beliefs. Setting $\rho = 1$ corresponds to the case of this appendix.

[3] For the explanation of this result, please see the proof of Proposition 3 in Appendix 4.2, which rigorously proves the corresponding property in the model with correct beliefs.

[4] This effect does not necessarily imply that the misbelief leads to a slower convergence because

When the initial reference points are too high, the players choose a level of cooperation lower than the steady state, like the model with correct beliefs. On $(\bar{z}', \bar{\pi}]$, eq. (2) implies $\pi_t \geq r_t$ is not sustainable; thus, $\pi_t < r_t$. This π_t satisfies

$$b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta}\right]\underline{\pi} - \frac{\delta\eta}{1-\delta}r_t = \frac{\pi_t - \eta(r_t - \pi_t)}{1-\delta}. \tag{4}$$

I prove $\pi_t < \bar{z}'$ by contradiction. Suppose $\pi_t \geq \bar{z}'$. Then, eq. (4) implies

$$\begin{aligned} b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta}\right]\underline{\pi} - \frac{\delta\eta}{1-\delta}\pi_t &= \frac{\pi_t}{1-\delta} - (1-\delta)\frac{\eta}{1-\delta}(r_t - \pi_t) \\ &< \frac{\pi_t}{1-\delta}. \end{aligned} \tag{5}$$

This inequality implies the existence of a steady state greater than π_t and, consequently, than $\bar{z}'$. This result contradicts the fact that $\bar{z}'$ is the unique steady state. Thus, $\pi_t < \bar{z}'$. Additionally, eq. (5) implies π_t decreases with r_t; a higher reference point hinders cooperation by providing the loss-evading incentive, like the model with correct beliefs.

4 Detailed Proofs of Lemma 6 and Proposition 3

This section provides the detailed proofs of Lemma 6 and Proposition 3.

4.1 Proof of Lemma 6

I prove Lemma 6 in three steps. First, I characterize the value function and the policy function of a modified problem that does not have the loss-evading incentive or the future cooperation losses. Second, I set up another modified problem in which only the loss-evading incentive is absent. I characterize the value function and the policy function, using the results of the first modified problem. Finally, using the results of

the steady state differs.

the second modified problem, I characterize the value function and the policy function of the original problem (5).

The first modified problem is given by

$$w_1^*(r_1) = \max_{\{\pi_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1}\pi_t, \tag{6}$$

$$s.t.\ \pi_t \in \Omega_1(r_t, w_1^*) \text{ for all } t,$$

$$r_{t+1} = (1-\rho)r_t + \rho\pi_t,$$

where

$$\Omega_1(r_t, w^*) = \left\{\pi_t \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \le w_1^*(r_t)\right\}.$$

This problem (6) does not have the loss utility terms with the indicator functions: $\mathbb{1}\{r_t > \pi_t\} \cdot \eta(r_t - \pi_t)$ in the objective function and $\mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta\,[r_t - b(\pi_t)]$ in the constraint of the feasible set. Problem (6) satisfies the corresponding functional equation:

$$w_1^*(r) = \max_{y \in \Omega_1(r, w_1^*)} \left[y + \delta w_1^*(\rho r + [1-\rho]y)\right].$$

I solved this problem similarly to the $[\underline{\pi}, \gamma]$ part of the proof of Proposition 5. I define X and $C(X)$ as

X :$[\underline{\pi}, \bar{\pi}]$,

$C(X)$:the set of bounded, continuous, and weakly increasing functions $f : X \to R$ with the sup norm that are weakly concave on X $s.t.\ \forall x \in X,\ \underline{\pi}'/(1-\delta) \le f(x) \le \bar{\pi}/(1-\delta)$.

On $C(X)$, I define the operator T by

$$Tf(x) = \max_{y \in \Omega_1(x,f)} \left[y + \delta f(\rho x + [1-\rho]y) \right],$$

where

$$\Omega_1(x,f) = \left\{ x \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq y + \delta f(\rho x + [1-\rho]y) \right\}.$$

Given function f, let the policy function, $g_{w1} : X \to Y$ be:

$$g_{w1}(x; f) = \operatorname*{argmax}_{y \in \Omega_1(x,f)} \; y + \delta f(\rho x + [1-\rho]y).$$

Since f is a weakly increasing function, $g_{w1}(x; f) = \max \Omega_1(x, f)$. I obtain the following properties following the same steps as Lemma 5,

1. $w_1^*(r)$ and $g_{w1}^*(r)$ are bounded and continuous on X.

2. $w_1^*(r)$ and $g_{w1}^*(r)$ are strictly increasing and strictly concave on X.

3. $r < g_{w1}^*(r) < \bar{z}$ on $[\underline{\pi}, \bar{z})$, $g_{w1}^*(\bar{z}) = \bar{z}$, and $\bar{z} < g_{w1}^*(r) < r$ on $(\underline{\pi}, \bar{z}]$ where $\bar{z}$ is implicitly defined as

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{z}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} \bar{z} = \bar{z} + \delta \frac{\bar{z}}{1-\delta}. \tag{7}$$

4. $r/(1-\delta) < w_1^*(r) < \bar{z}/(1-\delta)$ on $[\underline{\pi}, \bar{z})$, $w_1^*(\bar{z}) = \bar{z}/(1-\delta)$, and $\bar{z}/(1-\delta) < w_1^*(r) < r/(1-\delta)$ on $(\underline{\pi}, \bar{z}]$.

Second, I analyze another modified problem.

$$w_2^*(r_1) = \max_{\{\pi_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \left[\pi_t - \mathbb{1}\{r_t > \pi_t\} \cdot \eta(r_t - \pi_t)\right], \tag{8}$$
$$s.t.\ \pi_t \in \Omega_2(r_t, w_2^*) \text{ for all } t,$$
$$r_{t+1} = (1-\rho)r_t + \rho\pi_t,$$

where

$$\Omega_2(r_t, w_2^*) = \left\{\pi_t \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \le \pi_t + \delta w_2^*(r_{t+1})\right\}.$$

This problem has loss utilities in the objective function, and, consequently, the present value of future cooperation $\delta w_2^*(r_{t+1})$ on the RHS of the constraint in Ω_2 reflects future loss utilities. However, it does not count the current loss utility, effectively eliminating the loss-evading incentive. I obtain Lemma 1.

Lemma 1. $w_2^*(r) = w_1^*(r)$ and $g_{w2}^*(r) = g_{w1}^*(r)$ on $[\underline{\pi}, \bar{z}]$. $w_2^*(r) = \bar{z}/(1-\delta) - \eta(r-\bar{z})/(1-\delta\rho)$ and $g_{w2}^*(r) = \bar{z}$ on $r \in [\underline{\pi}, \bar{z}]$.

Proof. The functional equation of problem (8) is given by

$$w_2^*(r) = \max_{y \in \Omega_2(r, w_2^*)} \left[y - \mathbb{1}\{r > y\} \cdot \eta(r-y) + \delta w_2^*(\rho r + [1-\rho]y)\right].$$

I verify that $w_2^*(r)$ of Lemma 1 satisfies this functional equation. Suppose that $w_2^*(r) = w_1^*(r)$ for all $r \le \bar{z}$, and $w_2^*(r) = \bar{z}/(1-\delta) - \eta(r-\bar{z})/(1-\delta\rho)$ for all $r > \bar{z}$. Then, the RHS of the functional equation increases in y for all $r \in [\underline{\pi}, \bar{\pi}]$, implying $g_{w2}^*(r) = \max \Omega_2(r, w_2^*)$. Notice $g_{w1}^*(r) \in \Omega_2(r, w_2^*)$ for $r \le \bar{z}$. Additionally, $\Omega_2(r, w_2^*) \subset \Omega_1(r, w_1^*)$ for all $r \in [\underline{\pi}, \bar{\pi}]$ because $w_2^*(r) < \bar{z}/(1-\delta) < w_1^*(r)$ for all $r > \bar{z}$. Thus, $g_{w1}^*(r) = \max \Omega_1(r, w_2^*) =$

$\max \Omega_2(r, w_2^*) = g_{w2}^*(r)$. For $r > \bar{z}$, the constraint of Ω_2 becomes

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r \\
&\qquad \leq y + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(\rho r + [1-\rho]y - \bar{z})\right] \\
\iff &\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} \bar{z} \\
&\qquad \leq y + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta(1-\rho)}{1-\delta\rho}(y - \bar{z})\right].
\end{aligned} \tag{9}$$

Equality (7) and inequality (9) imply $\bar{z} \in \Omega_2(r, w_2^*)$. Suppose there exists $y > \bar{z}$ such that $y \in \Omega(r, w_2^*)$. Then, it follows from inequality (9) that

$$\begin{aligned}
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} y &\leq y + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(y - \bar{z})\right] \\
&\leq y + \delta w_1^*(y).
\end{aligned}$$

The last inequality implies $y \in \Omega_1(y, w_1^*)$, which contradicts $g_{w1}^*(r) = \max \Omega_1(r, w_1^*) < r$ for all $r > \bar{z}$. Thus, given $r > \bar{z}$, $y \notin \Omega_2(r, w_2^*)$ for all $y > \bar{z}$, implying $g_{w2}^*(r) = \max \Omega_2(r; w_2^*) = \bar{z}$ and $\rho r + [1-\rho] g_{w2}^*(r) > \bar{z}$. These results of $g_{w2}^*(r)$ verify that $w_2^*(r) = g_{w2}^*(r) + \delta w_2^*(\rho r + (1-\rho) g_{w2}^*(r)) = w_1^*(r)$ for all $r \leq \bar{z}$, and $w_2^*(r) = g_{w2}^*(r) - \eta(r - \bar{z}) + \delta w_2^*(\rho r + (1-\rho) g_{w2}^*(r)) = \bar{z}/(1-\delta) - \eta(r-\bar{z})/(1-\delta\rho)$ for all $r > \bar{z}$. □

Finally, I compare v^* to w_2^*. The only difference between problem (4) and the modified problem (8) is that the constraint of the second modified problem (8) does not have $\mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta\,[r_t - b(\pi_t)] \geq 0$ as a short-term gain. Suppose that for some $r \in [\underline{\pi}, \bar{\pi}]$ $v^*(r) > w_2^*(r)$ and let $\{\pi_t^o\}_{t=1}^\infty$ be the optimized path with $r_1 = r$ $(v^*(r_1) = \sum_{t=1}^\infty \delta^{t-1}\,[\pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o)])$. This $\{\pi_t^o\}_{t=1}^\infty$ is feasible in the second modified problem (8) with $w_2^*(r_2) = \sum_{t=2}^\infty \delta^{t-1}\,[\pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o)]$ in the constraint, which implies $w_2^*(r_t) \geq v^*(r_t)$. This result contradicts $v^*(r_t) > w_2^*(r_t)$. Thus, $v^*(r_t) \leq w_2^*(r_t)$ for all $r_t \in [\underline{\pi}, \bar{\pi}]$. It immediately follows that $g^*(r_t) = g_{w2}^*(r_t)$ for $r_t \leq \bar{z}$, implying

$v^*(r) = w_2^*(r)$. For $r_t \geq \bar{z}$, suppose $g^*(r_t) \geq \bar{z}$. Then, it follows that

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g^*(r_t)) - \mathbb{1}\{r > b(g^*(r_t))\} \cdot \eta \left[r - b(g^*(r_t))\right] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t$$
$$\leq v^*(r_t)$$
$$\Longrightarrow \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g^*(r_t)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \leq w_2(r_t).$$

This inequality contradicts $y \notin \Omega_2(r, w_2^*)$ for all $r, y > \bar{z}$. Thus, $g^*(r) < \bar{z}$. This completes the proof of Lemma 6.

4.2 Proof of Proposition 3

I define X and $C(X)$ as:

$$\begin{aligned} X :& [\underline{\pi}, \bar{\pi}] \\ C(X) :& \text{the set of bounded and continuous functions } f : X \to R \\ & \text{with the sup norm that are weakly concave on } [\bar{z}, \bar{\pi}]. \\ & s.t.\ \forall x \in [\underline{\pi}, \bar{z}],\ f(x) = w_1^*(x), \\ & \quad \forall x \in [\bar{z}, \bar{\pi}],\ f(x) \leq \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}), \\ & \quad \forall x, \forall y \in [\bar{z}, \bar{\pi}]\ s.t.\ x < y,\ \frac{f(y) - f(x)}{y - x} \geq -\frac{1+\eta}{\delta(1-\rho)}, \end{aligned}$$

where w_1^* is that of Appendix 4.1. On $C(X)$, I define the operator T by

$$Tf(x) = \max_{y \in \Gamma(x,f)} \left[y - \mathbb{1}\{x > y\} \cdot \eta(x - y) + \delta f(\rho x + [1-\rho]y)\right],$$

where

$$\Gamma(x,f)=\Big\{x\in[\underline{\pi},\bar{\pi}]:\left[1-\frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]b(y)+\mathbb{1}\{x>y\}\cdot\eta\left[\min\{x,b(y)\}-y\right]$$
$$+\left[\frac{\delta}{1-\delta}+\frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi}-\frac{\delta\eta\rho}{1-\delta\rho}x\leq y+\delta f(\rho x+[1-\rho]y)\Big\}.$$

Given function f, I let the policy function, $g:X\to Y$ be:

$$g(x;f)=\underset{y\in\Gamma(x;f)}{\operatorname{argmax}}\ y-\mathbb{1}\{x>y\}\cdot\eta(x-y)+\delta f(\rho x+[1-\rho]y).$$

I denote $g(x;f)$ by $g(x)$ unless it is confusing.

$g(x;f)=\max\Gamma(x,f)$ Suppose $g(x)<\max\Gamma(x,f)=\gamma(x,f)$ for some $x\in(\bar{z},\bar{\pi}]$. Then,

$$\begin{aligned}0<&\{g(x)-\eta(x-g(x))+\delta f(\rho x+[1-\rho]g(x))\}-\{\gamma(x)-\eta(x-\gamma(x))+\delta f(\rho x+[1-\rho]\gamma(x))\}\\<&(1+\eta)\left[g(x)-\gamma(x)\right]+\delta(1-\rho)\left[g(x)-\gamma(x)\right]\left[-\frac{1+\eta}{\delta(1-\rho)}\right]=0.\end{aligned}$$

This is a contradiction. Thus, $g(x)=\max\Gamma(x,f)$.

$f(x)=Tf(x)=w_1^*(x)$ for all $x\leq\bar{z}$ $w_1^*(x)>\bar{z}/(1-\delta)>f(x)$ on $(\bar{z},\bar{\pi}]$ implies $\Gamma(x,f)\subset\Omega_1(x,w_1^*)$. Additionally, we have $g_{w1}^*(x)=\max\Omega_1(x,f)\in\Gamma(x,f)$. Thus, $g^*(x,f)=g_{w1}^*(x)$ on $[\underline{\pi},\bar{z}]$.

$g(x;f)<\bar{z}$ for $x>\bar{z}$ Given $x>\bar{z}$, suppose y$\in\Gamma(x,f)$ for some $y\geq\bar{z}$. Then, given w_2^* in Lemma ?? of Appendix 4.1, $f(r)=w_2^*(r)$ for $r\leq\bar{z}$ and $f(r)\leq w_2^*(r)$ for $r>\bar{z}$, implying

$$\left[1-\frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]b(y)+\left[\frac{\delta}{1-\delta}+\frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi}-\frac{\delta\eta\rho}{1-\delta\rho}x\leq y+\delta w_2^*(\rho x+[1-\rho]y).$$

This inequality contradicts that given $x>\bar{z}$, $y\notin\Omega_2(x,w_2^*)$ for all $y>\bar{z}$, obtained in Appendix (4.1). Thus, given $x>\bar{z}$, $\max\Gamma(x,f)\leq\bar{z}$. Next, suppose $\bar{z}\in\Gamma(x,f)$ for some

$x > \bar{z}$.

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{z}) + \mathbb{1}\{x > \bar{z}\} \cdot \eta \left[\min\{x, b(\bar{z})\} - \bar{z}\right]$$
$$+ \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq \bar{z} + \delta f(\rho x + [1-\rho]\bar{z})$$
$$\implies \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{z}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} \bar{z} < \bar{z} + \delta \frac{\bar{z}}{1-\delta}.$$

This contradicts eq. (7). Thus, $\bar{z} \notin \Gamma(x; f)$ for all $x > \bar{z}$. Therefore, $g(x; f) < \bar{z}$ for $x > \bar{z}$.

Continuities The continuity of f, y$\notin \Gamma(x, f)$ for all $y \geq \bar{z}$ and $x > \bar{z}$, and $g(x) = \max \Gamma(x, f)$, imply that $g(x)$ satisfies

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x)) + \eta(\min\{x, b(g(x))\} - g(x)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \quad (10)$$
$$= g(x) + \delta f(\rho x + (1-\rho) g(x)).$$

The continuities of Tf and g follow in the same way as Case 1.

Given Assumption (5), $b(g(x)) > x$ $b(g(x)) - x$ is continuous, and $b(g(\bar{z})) - \bar{z} = b(\bar{z}) - \bar{z} > 0$. Consequently, if $\bar{\pi}$ is sufficiently close to $\bar{z}$, $b(g(x)) > x$ for all $x \in [\bar{z}, \bar{\pi}]$.

$Tf(x) < \bar{z}/(1-\delta) - \eta(x - \bar{z})/(1-\delta\rho)$ for $x > \bar{z}$ Subsequently, given $x > \bar{z}$,

$$\begin{aligned}
Tf(x) &= g(x; f) - \mathbb{1}\{x > g(x; f)\} \cdot \eta(x - g(x; f)) + \delta f(\rho x + [1-\rho] g(x; f)) \\
&= g(x; f) - \eta(x - g(x; f)) + \delta f(\rho x + [1-\rho] g(x; f)) \\
&< \bar{z} - \eta(x - \bar{z}) + \delta f(\rho x + (1-\rho)\bar{z}) \\
&\leq \bar{z} - \eta(x - \bar{z}) + \delta \left[\frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(\rho x + (1-\rho)\bar{z} - \bar{z})\right] \\
&= \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}),
\end{aligned}$$

where the first inequality follows from that $[f(y) - f(x)]/(y - x) \geq -\eta/[\delta(1-\rho)]$ for all x and y *s.t.* $\bar{z} \leq x < y \leq \bar{\pi}$ and that $[f(y) - f(x)]/(y - x) > 0$ for all x and y *s.t.* $\underline{\pi} \leq x < y \leq \bar{z}$.

Concavity Given that f is increasing and concave on $[\underline{\pi}, \bar{z}]$ and that f is decreasing and concave on $[\bar{z}, \bar{\pi}]$, f is weakly concave on $[\underline{\pi}, \bar{\pi}]$. The rest is similar to Case 1. Given $x_1, x_2 \geq \bar{z}$,

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{[1-\theta]b(g(x_1)) + \theta b(g(x_2))\} + \eta(([1-\theta]x_1 + \theta x_2) - (1-\theta)g(x_1) - \theta g(x_2)) \\
&\quad + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}([1-\theta]x_1 + \theta x_2) \\
&= (1-\theta)g(x_1) + \theta g(x_2) + (1-\theta)\,\delta f(\rho x_1 + [1-\rho]g(x_1)) + \theta\delta f(\rho x_2 + [1-\rho]g(x_2)).
\end{aligned} \tag{11}$$

where $\theta \in (0, 1)$. Subsequently, eq.(11), the convexity of b, and the weak concavity of f imply

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{b([1-\theta]g(x_1) + \theta g(x_2))\} + \eta\left(\tilde{x} - [(1-\theta)g(x_1) + \theta g(x_2)]\right) \\
&+ \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\tilde{x} \qquad (12) \\
&\leq (1-\theta)g(x_1) + \theta g(x_2) + \delta f\left(\rho\tilde{x} + (1-\rho)\{[1-\theta]g(x_1) + \theta g(x_2)\}\right), \qquad (13)
\end{aligned}$$

where $\tilde{x} = [1-\theta]x_1 + \theta x_2$. It follows from eq. (10) that:

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{b(g(\tilde{x}))\} + \eta(\tilde{x} - g(\tilde{x})) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\tilde{x} \\
&= g(\tilde{x}) + \delta f(\rho\tilde{x} + (1-\rho)g(\tilde{x})).
\end{aligned} \tag{14}$$

Since $g(\tilde{x}) = \max\Gamma(x, f)$, eq. (13) and (14) imply

$$(1-\theta)g(x_1) + \theta g(x_2) \leq g(\tilde{x}) = g([1-\theta]x_1 + \theta x_2).$$

Thus, $g(x)$ is weakly concave. Consequently, so is $Tf(x)$. The concavity, $g(\bar{z}) = \bar{z}$, and $g(x) < \bar{z}$ imply $g(x)$ is strictly decreasing on $[\bar{z}, \bar{\pi}]$. In turn, this implies $g(x_1) \neq g(x_2)$ for $x_1 \neq x_2 > \bar{z}$ and repeating the same procedure above yields the strict concavity of $g(x)$ on $[\bar{z}, \bar{\pi}]$. The same process with $x_1, x_2 \leq \bar{z}$ shows the strict concavity of $g(x)$ on $[\underline{\pi}, \bar{z}]$. Further, as we know $g(x)$ is increasing on $[\underline{\pi}, \bar{z}]$ and decreasing on $[\bar{z}, \bar{\pi}]$, $g(x)$ is strict concave on $[\underline{\pi}, \bar{\pi}]$. Thus, given that (i) f is increasing and strictly concave on $[\underline{\pi}, \bar{z}]$, (ii)decreasing and weakly concave on $[\underline{\pi}, \bar{\pi}]$, and (iii) $g(x)$ is strictly concave, Tf is strictly concave on $[\underline{\pi}, \bar{\pi}]$. Additionally, $[f(y) - f(x)]/(y-x) \geq -(1+\eta)/\delta(1-\rho)$ and $g(x)$ decreasing imply $Tf(x)$ decreasing on $[\underline{\pi}, \bar{z}]$.

The slope of Tf Given $x_1, x_2 > \bar{z}$ such that $x_2 < x_1$, $g(x_2) > g(x_1)$. Eq. (10) for x_1 and $g(x_2) \notin \Gamma(x_1, f)$ imply

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_1)) + \eta(x_1 - g(x_1)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x_1 \\
&\qquad = g(x_1) + \delta f(\rho x_1 + [1-\rho]g(x_1)), \\
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_2)) + \eta(x_1 - g(x_2)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x_1 \\
&\qquad > g(x_2) + \delta f(\rho x_1 + [1-\rho]g(x_2)).
\end{aligned}$$

Manipulating these yields

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\frac{b(g(x_2)) - b(g(x_1))}{g(x_2) - g(x_1)} - 1 - \eta - \delta\frac{f(\rho x + [1-\rho]g(x_2)) - f(\rho x + [1-\rho]g(x_1))}{g(x_2) - g(x_1)} \tag{15}$$

> 0.

Eq. (10) for x_1 and x_2 also imply

$$\left[1-\frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\cdot\frac{b(g(x_1))-b(g(x_2))}{g(x_1)-g(x_2)}\cdot\frac{g(x_1)-g(x_2)}{x_1-x_2}-\frac{\delta\eta\rho}{1-\delta\rho}$$
$$=(1+\eta)\frac{g(x_1)-g(x_2)}{x_1-x_2}-\eta$$
$$+\delta(1-\rho)\frac{f(\rho x_1+(1-\rho)g(x_1))-f(\rho x_1+(1-\rho)g(x_2))}{(1-\rho)g(x_1)-(1-\rho)g(x_2)}\frac{g(x_1)-g(x_2)}{x_1-x_2}$$
$$+\delta\rho\frac{f(\rho x_1+(1-\rho)g(x_2))-f(\rho x_2+(1-\rho)g(x_2))}{\rho x_1-\rho x_2}.$$

This becomes:

$$\frac{g(x_1)-g(x_2)}{x_1-x_2} \tag{16}$$
$$=\frac{-\eta(1-2\delta\rho)/(1-\delta\rho)+\delta\rho F_1}{[1-\delta\eta(1-\rho)/(1-\delta\rho)]\,[b(g(x_1))-b(g(x_2))]\,/\,[g(x_1)-g(x_2)]-1-\eta-\delta(1-\rho)F_2}$$

where

$$F_1=\frac{f(\rho x_1+(1-\rho)g(x_2))-f(\rho x_2+(1-\rho)g(x_2))}{\rho x_1-\rho x_2},$$
$$F_2=\frac{f(\rho x_1+(1-\rho)g(x_1))-f(\rho x_1+(1-\rho)g(x_2))}{(1-\rho)g(x_1)-(1-\rho)g(x_2)}.$$

We know that the left-side hand of eq. (16) is negative and that the denominator of the right-hand side is positive from eq. (15).[5] Since $F_2\geq\min\{0,-(1+\eta)/\delta(1-\rho)\}=-(1+\eta)/\delta(1-\rho)$, eq. (16) implies

$$\frac{g(x_1)-g(x_2)}{x_1-x_2}\leq\frac{-\eta(1-2\delta\rho)/(1-\delta\rho)+\delta\rho F_1}{[1-\delta\eta(1-\rho)/(1-\delta\rho)]}\frac{g(x_1)-g(x_2)}{b(g(x_1))-b(g(x_2))}. \tag{17}$$

[5]This implies that $\delta\rho>1/2$ is a sufficient condition for $\rho x+(1-\rho)g(x)$ to be greater than $\bar{z}$ for all $x>\bar{z}$ because $\delta\rho>1/2$ requires $F_1<0$.

Finally,

$$\begin{aligned}
Tf(x_1) - Tf(x_2) =& g(x_1) - \eta\left[x_1 - g(x_1)\right] + \delta f(\rho x_1 + [1-\rho]g(x_1)) \\
& - \left\{g(x_2) - \eta\left[x_2 - g(x_2)\right] + \delta f(\rho x_2 + [1-\rho]g(x_2))\right\} \\
=& \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\frac{b(g(x_1)) - b(g(x_2))}{g(x_1)-g(x_2)}\frac{g(x_1)-g(x_2)}{x_1 - x_2}(x_1 - x_2) - \frac{\delta\eta\rho}{1-\delta\rho}(x_1 - x_2) \\
\geq& \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\frac{b(g(x_1)) - b(g(x_2))}{g(x_1)-g(x_2)}\left[\frac{-\eta(1-2\delta\rho)/(1-\delta\rho) + \delta\rho F_1}{[1-\delta\eta(1-\rho)/(1-\delta\rho)]}\right] \\
& \cdot \frac{g(x_1)-g(x_2)}{b(g(x_1)) - b(g(x_2))}(x_1 - x_2) - \frac{\delta\eta\rho}{1-\delta\rho}(x_1 - x_2) \\
=& \left[-\eta + \delta\rho F_1\right](x_1 - x_2)
\end{aligned}$$

where the second equality follows from eq. (10) and the first inequality follows from eq. (17) and $b(\cdot)$ being increasing. If $\rho x_2 + (1-\rho)g(x_2) \leq \bar{z}$,

$$-\eta + \delta\rho F_1 > -\eta > -\frac{1+\eta}{\delta(1-\rho)}.$$

If $\rho x_2 + (1-\rho)g(x_2) \leq \bar{z}$,

$$-\eta + \delta\rho F_1 > -\eta - \delta\frac{\rho(1+\eta)}{\delta(1-\rho)} = -\frac{\rho+\eta}{1-\rho} > -\frac{1+\eta}{\delta(1-\rho)}$$

In either case, $[Tf(x_1) - Tf(x_2)]/(x_1 - x_2) > -(1+\eta)/\delta(1-\rho)$.

Convergence Given the results above, $T : C(X) \to C'(X)$ where $C'(x)$ is defined as:

$$
\begin{aligned}
X :& [\underline{\pi}, \bar{\pi}] \\
C'(X) :& \text{the set of bounded, continuous, and strictly concave functions } f : X \to R \\
& \text{with the sup norm that are strickly decreasing on } [\bar{z}, \bar{\pi}] \\
& s.t.\ \forall x \in [\underline{\pi}, \bar{z}],\ f(x) = w^*(x) \\
& \quad \forall x \in [\bar{z}, \bar{\pi}],\ f(x) < \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}) \\
& \quad \forall x, \forall y \in [\bar{z}, \bar{\pi}]\ s.t.\ x < y,\ \frac{f(y) - f(x)}{y - x} > -\frac{1+\eta}{\delta(1-\rho)}
\end{aligned}
$$

which is a subset of $C(X)$. Using this operator T, let

$$
\begin{cases}
f_1(x) & = \begin{cases} w^*(x) & x \in [\underline{\pi}, \bar{z}] \\ \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}) & x \in [\bar{z}, \bar{\pi}] \end{cases} \\
f_{n+1} & = T f_n \quad \text{for} \quad n \in \mathbb{N}
\end{cases}
$$

be a sequence of functions produced by T. f_1 is in $C(X)$. It follows that

$$
\begin{cases}
f_2(x) = f_1(x) = w^*(x) & x \in [\underline{\pi}, \bar{z}] \\
f_2(x) < f_1(x) & x \in (\bar{z}, \bar{\pi}]
\end{cases}
$$

Given the strict concavity of f_n on $[\underline{\pi}, \bar{\pi}]$, the rest of the proof follows the same steps as Case 1. This completes the proof of Proposition 3.