Gradual Development and Rough Transition of Cooperation with Reference Dependence*†

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Abstract

I study cooperation among reference-dependent and loss-averse players in a dynamic game with complete information. Every period, players choose their cooperation levels that determine their intrinsic payoffs and update their reference points in a backward-looking way. I characterize the subgame-perfect equilibrium that maximizes the utility with Nash reversion and show that the development of cooperation exhibits gradualism. After initiating cooperation, players experience higher payoffs than their initial reference points. Consequently, the reference points rise, making a penalty more severe for a deviation and enabling further cooperation. This paper additionally illustrates how the developed cooperation responds to a

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structural shock. When a steady-state cooperation level shifts down, transitioning to the new level entails a loss. This loss generates an additional deviation incentive, and the players undergo cooperation lower than the new steady-state level.

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1 Introduction

Standard models explain cooperation among non-cooperative players as an equilibrium outcome in which players rationally weigh their gains and losses by a deviation from cooperation. However, players are not always rational in the real world; as one of the established patterns of irrationality, they are reference-dependent and loss-averse (Kahneman and Tversky (1979)). They evaluate payoffs relative to their reference points and exhibit a strong aversion to losses—payoffs lower than the reference point. Given losses' central role in the models of cooperation, the implications of reference dependence are crucial in our understanding of cooperative behaviors. How differently do reference-dependent players forge cooperation from rational players? What phenomena in cooperation, if any, can reference dependence explain?

This paper achieves two things. First, it shows that reference-dependent and loss-averse players develop their cooperation gradually, which appears to be gradualism or "starting small," which is prevalent in cooperative relationships. Second, it illustrates how the developed cooperation responds when the economic environment changes. The latter analysis shows, among other things, that when the players have to scale back their cooperation in a new steady state, they have to undergo a cooperation level lower than the new steady-state level during the transition.

I investigate an infinite horizon dynamic game with complete information in which players have reference points. Symmetric players choose their cooperation levels individually and simultaneously every period. Their flow utility consists of an intrinsic payoff and a loss utility. The intrinsic payoff is a function of all the players' levels of cooperation and exhibits prisoner's-dilemma-type properties; the first-best cooperation is infeasible in a one-shot game because a deviation is lucrative, and a static Nash equilibrium yields low cooperation. The loss utility becomes non-zero and negative only if the intrinsic payoff is below the player's reference point, reflecting loss aversion. The reference point is backward-looking, as DellaVigna et al. (2017) and Bowman et al. (1999). It is updated every period as the weighted average of the previous value and the latest intrinsic payoff following Bowman et al. (1999). Throughout the paper, I consider subgame-perfect equilibria in which the players employ a Nash-reversion strategy. When a player deviates from a cooperative level of cooperation, the other players impose a static Nash equilibrium that generates low intrinsic payoffs in the following periods as a penalty. As is well known, there are infinite numbers of subgame-perfect equilibria in this type of model. I focus on the one that maximizes the players' present value of utility. The model inherits from a standard model a tension between the short-term gain and the long-term loss from a deviation; a Nash-reversion strategy can make cooperation in an equilibrium higher than the static Nash level, but when the discount factor is not sufficiently large, it cannot reach the first-best level. I assume that the players cannot achieve the first-best level when their reference points are sufficiently low such as the Nash-level payoff, and study how the cooperative level evolves.

Reference dependence generates three forces that influence the development and transitions of cooperation. The first is an aversion to losing cooperation, the only force at work when the players start cooperation with a low initial reference point. In my model, the long-term loss for a deviation contains loss utilities, generating the aversion to losing cooperation. Given intrinsic flow utilities that a penalty inflicts, a higher reference point exacerbates the losses, intensifying the aversion.

Cooperation develops gradually as the aversion to losing cooperation strengthens over time. Players initiating cooperation experience payoffs higher than their reference points. Consequently, the reference points rise, reinforcing the aversion; in other words, the players become used to high payoffs and become more averse to losing them. This strengthening aversion enables the players to cooperate further in the next period. In turn, this elevates their reference points and strengthens the aversion further. Repeating this process results in a gradual development of cooperation, and their cooperation eventually converges to a steady state.

The developed cooperation is exposed to two additional forces when the economic environment unfavorably changes. Consider players who have fostered cooperation to the point where the intrinsic payoff and their reference points have reached a steady state. Suppose a structural shock shifts down the intrinsic payoff from given cooperation. In that case, the players can no longer maintain the payoff at the original steady state because it requires higher cooperation than before. Consequently, the players incur a loss utility even when they implement a cooperative level of cooperation after the shock, which generates the two forces. The first is the loss-evading deviation incentive. A loss by a cooperative level of cooperation incentivizes the players to deviate to avoid it, augmenting the short-term gain of a deviation. The second force is the diminishing future cooperation value. When the transition to the new steady state takes multiple periods and entails loss utilities in future, the value of the future cooperative levels of cooperation reduces. Consequently, the long-term loss of a deviation decreases. These two effects counteract the aversion to losing cooperation.

The net effect necessitates that players undergo a cooperation level lower than the new steady-state level. A reference point higher than the new steady state hinders them from cooperation. Only after experiencing payoffs lower than the new steady state level and lowering their reference points can they reach the new steady state. The intuition behind this result is that those living in past glories are not trustable because they try to cling to them by betraying others; it is not until they become used to the new and less pleasant reality that they can be trustable. This analysis highlights that not only the history of actions but also the reference point, or equivalently the history of outcomes (intrinsic payoffs), shape players' reputations or trustability in this game.

To the best of my knowledge, this paper is the first to demonstrate that reference dependence and loss aversion generate gradualism, or "starting small," in cooperation. Unlike my model, most existing works on gradualism in relationship features incomplete information. They analyze situations in which players are uncertain about their partners' types (Sobel 1985; Watson 1999, 2002; Rauch and Watson 2003; Hua and Watson 2022; Furusawa and Kawakami 2008) and, additionally, in some cases, players can match with other partners after relationship termination (Ghosh and Ray 1996; Kranton 1996).[1] Research on gradualism in games with complete information focus on other factors, such as the irreversibility of cooperation (Lockwood and Thomas 2002).

The results of this work apply also to gradualism in other fields such as trade policy. The literature analyzes trade liberalization with incomplete tariff elimination as an equilibrium result of a tariff-setting game between strategic governments in an infinite time horizon. The welfare function that the governments maximize falls into a class of intrinsic payoff functions in my model as Appendix A.[2] While there have been several explanations of gradualism (Staiger 1994; Devereux 1997; Furusawa and Lai 1999; Bond and Park 2002; Maggi and Rodriguez-Clare 2007; Zissimos 2007), there is no work with reference dependence as the driver of gradualism in a strategic setting. Although reference dependence or loss aversion at an aggregate level is not as established a concept as the individual level, efforts theoretically and empirically investigate behavioral effects in trade policy (Bernardes 2003; Freund and Özden 2008; Tovar 2009).

The implication for transitional dynamics is a unique contribution. From collusion by firms to trade agreements to gang memberships, no cooperation is immune to changes in their economic environments like a new whistleblower regulation, growing trade protectionism, or stricter law enforcement. However, analysis of the transitional dynamics is scant in the literature. A mature relationship has resolved the problem of incomplete information; therefore, models of cooperation development with incomplete information typically imply an instant transition from a steady state to a new one. My model provides rich implications for transitional dynamics of mature relationships and cooperation.

[1] Datta (1996) also studies gradualism in a model with random matching but with complete information.

[2] The welfare for a government to maximize is the sum of consumer surplus, producer surplus, and tariff revenues. When a government redistributes an increase in this welfare among its citizens, the welfare is the sum of the gains from tariff reductions for individual citizens. If citizens are reference-dependent and loss-averse, their individual loss utilities aggregate a loss utility at the government level.

The rest of this paper is structured as follows. Section 2 introduces the model. Section 3 examines Nash-reversion strategies in this model. Section 4 studies the conditions for subgame-perfect equilibria with the Nash-reversion strategies. Section 5 solves for subgame-perfect equilibrium that maximizes the present value of utility with a Nash-reversion strategy and illustrates the gradual development of cooperation. Section 6 analyzes how the cooperation level transitions to a new steady state when the economic environment changes. Section 7 concludes.

2 Setup

I study a dynamic game with complete information by N symmetric players. I denote the set of players by $\boldsymbol{N}$. There are infinite periods, which I denote by t. In each period, the players individually and simultaneously choose their cooperation levels. I denote player i's cooperation level in period t by $\alpha_{i,t} \in A$ where $A = [0, \bar{\alpha}]$ and the vector of cooperation levels in period t by $\boldsymbol{\alpha}_t = (\alpha_{1t}, \alpha_{2t}, \ldots, \alpha_{Nt})$.

Player i's flow utility consists of an intrinsic payoff $\pi_i(\boldsymbol{\alpha}_t)$ and a loss utility $\ell_i(\boldsymbol{\alpha}_t|r_{it})$. Given the reference point r_{it} of player i in period t, it is given by

$$u(\boldsymbol{\alpha}_t|r_{i,t}) = \pi_i(\boldsymbol{\alpha}_t) + \ell_i(\boldsymbol{\alpha}_t|r_{it})$$

where

$$\ell_i(\boldsymbol{\alpha}_t|r_{it}) = \begin{cases} 0 & \text{if } \pi(\boldsymbol{\alpha}_t) \geq r_{i,t} \\ -\eta\left[r_{it} - \pi_i(\boldsymbol{\alpha}_t)\right] & \text{if } \pi(\boldsymbol{\alpha}_t) < r_{i,t} \end{cases}$$

and $0 < \eta \leq 1$. The intrinsic payoff is symmetric among the players in the sense $\pi_i(\boldsymbol{\alpha}_t) = \pi_j(\boldsymbol{\alpha}'_t)$ when $\boldsymbol{\alpha}'_t = (\alpha'_{1t}, \alpha'_{2t}, \ldots, \alpha'_{Nt})$, $\alpha_{it} = \alpha'_{jt}$, $\alpha_{jt} = \alpha'_{it}$, and $\forall k \in \boldsymbol{N} \backslash i, j$, $\alpha_{kt} = \alpha'_{kt}$. The loss utility term $\ell_i(\boldsymbol{\alpha}_t|r_{it})$ captures the reference dependence and loss aversion. It measures losses in terms of intrinsic payoffs as Kőszegi and Rabin (2006) and others; when the intrinsic payoff $\pi_i(\boldsymbol{\alpha}_t)$ is below the reference point r_{it}, she perceives it as a loss and incurs the loss utility, $-\eta \cdot [r_{i,t} - \pi_i(\boldsymbol{\alpha}_t)] < 0$. The parameter η measures the degree of loss aversion, and if $\eta = 0$ instead of $0 < \eta \leq 1$, the model becomes a standard one without reference dependence.

There are simplifying assumptions in this functional form. First, the flow utility u has only the loss utility, not both loss and gain utilities.[3] It disables us to weigh the importance of gain-loss utilities relative to intrinsic payoff separately from loss utility relative to gain utility. However, it retains the property of loss aversion and simplifies the algebra. Also, in structural estimations of a reference-dependent job search model, DellaVigna et al. (2017) report that a model only with loss utility fits the data similarly to a model with both gain and loss utilities. Thus, using only the loss utility for tractability is a reasonable choice that is not very harmful. Second, it abstracts from the diminishing sensitivity of the prospect theory (Kahneman and Tversky 1979); the loss utility is piece-wise linear in loss ($r_{it} - \pi_i(\boldsymbol{\alpha}_t)$). This functional form enables us to study the main idea of this paper in a relatively simple model. I set the interval of η value such that it is roughly consistent with that the sensitivity to a loss is about twice as large as that to the same amount of gain (Tversky and Kahneman 1991). The players discount future flow utilities by a discount factor $\delta \in (0, 1)$ and maximizes the present value of their utility $\sum_{t=0}^{\infty} \delta^t u(\boldsymbol{\alpha}_t | r_{i,t})$.

I assume several properties of the intrinsic payoff $\pi_i(\boldsymbol{\alpha}_t)$ that can be micro-founded by either collusion in Cournot competition or trade liberalization as explained in Appendix A. $\pi_i(\boldsymbol{\alpha}_t)$ is continuous in α_j and twice differentiable with respect to α_j for all $j \in N$. In this paper, I focus on equilibria where the players symmetrically choose the same level of cooperation. To simplify the notations, I define $\tilde{\pi}(\alpha)$ as Definition 1. I assume $\tilde{\pi}$ is increasing as 1.

Definition 1 (Symmetric Cooperation)**.** $\tilde{\pi}(\alpha) = \pi_i(\boldsymbol{\alpha}_t)$ where $\alpha_{it} = \alpha$ for all $i \in N$.

Assumption 1. $\tilde{\pi}(\alpha)$ *strictly increases in* α*.*

I let $\bar{\pi} = \max_{\alpha \in A} \tilde{\pi}(\alpha) = \tilde{\pi}(\bar{\alpha})$ and refer to $\bar{\alpha}$ as the *first-best cooperation*. Assumption 1 ensures

[3] Other works that employ this formulation includes Freund and Özden (2008).

that the inverse function $\tilde{\pi}^{-1}(\pi)$ exists. Using this inverse function, I define the best-response intrinsic payoff b as a function of π.[4]

Definition 2 (Best-Response)**.** $b(\pi) = \max_{\alpha_i \in A} \pi_i(\alpha_i, \tilde{\pi}^{-1}(\pi), \ldots, \tilde{\pi}^{-1}(\pi))$.

I assume two properties for the best-response intrinsic payoff as Assumption 2.

Assumption 2. *b is convex, and there exists unique $\underline{\pi} \in [\tilde{\pi}(0), \bar{\pi})$ such that $b(\underline{\pi}) = \underline{\pi}$.*

Consider a one-shot game without a loss utility ($\eta = 0$). Symmetric Cooperation in such equilibrium cannot achieve the maximum intrinsic payoff $\bar{\pi}$ because Assumption 2 implies that $\underline{\pi}$ is the intrinsic payoff in a unique symmetric pure-strategy Nash equilibrium. I denote the cooperation level that produces $\underline{\pi}$ by α^N ($\alpha^N = \tilde{\pi}^{-1}(\underline{\pi})$). Also, I refer to α^N as the *Nash cooperation* in the rest of the paper, although whether $\underline{\pi}$ is the intrinsic payoff in a Nash equilibrium with a loss utility ($\eta > 0$) has not yet been examined. In Section 3, I show that $\underline{\pi}$ is indeed the intrinsic payoff in a subgame-perfect equilibrium even with a loss utility ($\eta > 0$).

The reference point is backward-looking and adaptive, following Bowman et al. (1999) and DellaVigna et al. (2017). Specifically, it is given by

$$r_{i,t} = \begin{cases} \rho r_{i,t-1} + (1-\rho)\pi_i(\boldsymbol{\alpha}_{t-1}) & \text{if } t \geq 2 \\ r_1 & \text{if } t = 1 \end{cases}$$

where $0 < \rho < 1$. The initial reference point $r_1 \in \mathbb{R}$ takes an exogenously given common value among the players, which is public information. The parameter ρ measures the persistence of a reference point. This backward-looking reference point is a crucial assumption in my model because it only rises once players experience high payoffs. In the structural estimations by DellaVigna et al. (2017), the backward-looking and adaptive reference point outperforms other types of reference points, including the forward-looking one (Kőszegi and Rabin 2006) and the status quo,

[4] $b(\pi)$ is the best response in terms of intrinsic payoff and in terms of flow utility.

which corresponds to $\rho = 0$ in my model, in fitting job search patterns.[5][6] Also, Post et al. (2008) report the persistence of reference points, although the functional form differs from mine. Thus, a backward-looking and adaptive reference point is a reasonable assumption.

3 Nash-Reversion Strategy

The analysis of this paper focuses on the pure-strategy subgame-perfect equilibria with Nash-reversion strategies, which use Nash equilibrium as the penalty (Friedman 1971). Consequently, the range of cooperation levels feasible in the subgame-perfect equilibria is generally limited compared to the optimal penalty (Abreu 1988). However, by focusing on a relatively simple form of penalty, this paper provides rich analytical results; it transparently illustrates how the reference-dependent utility plays a role in the level of cooperation and how it can vary over time in subgame-perfect equilibria.

This section examines the Nash-reversion strategies and the best deviation when others employ a Nash-reversion strategy. We must first know where to revert–a penalty. A natural candidate is a Nash strategy s^N in Definition 3.

Definition 3. A Nash strategy s^N is a sequence of mapping σ_t^N that maps any history of cooperations $\{\boldsymbol{\alpha}_k\}_{k=1}^{t-1} \in A^{N(t-1)}$ and the initial reference point $r_1 \in \mathbb{R}$ into the Nash cooperation α^N. That is, $\forall t \in \mathbb{N}, \sigma_t^N : A^{N(t-1)} \times \mathbb{R} \mapsto \alpha^N$.

We cannot take it for granted that all the players employing s^N, $(s^N, s^N, \ldots s^N)$, forms a subgame-perfect equilibrium in this game. It is so for two reasons. First, the flow utility $u(\boldsymbol{\alpha}_t|r_{i,t})$ contains

[5]DellaVigna et al. (2017) use the average income over the N previous periods in their benchmark model instead of AR(1) formation like mine for computational reasons. They report that the fits of these two reference points are similar.

[6]In addition to the empirical support, there is another reason why I do not allow ρ to be one. $\rho = 1$ disconnects a reference point before a deviation and the loss utilities by a penalty. If $\rho = 1$, the reference point in the first penalty period is the intrinsic utility by the best deviation, which I show independent of the reference point in Section 3. Given this, the magnitude of the aversion to losing cooperation would become independent of the reference point, and gradualism does not emerge.

the loss-aversion term $\ell(\boldsymbol{\alpha}_t|r_{i,t})$. Second, this model has a dynamic effect. An action in a given period affects the reference points in the following periods, and, therefore, the maximizer of the flow utility u possibly differs from that of the present value of utility ($\sum_{k=t}^{\infty} \delta^{k-t} u_{i,k}$), even when the other players choose the same action every period, regardless of the history of the game. To see these effects clearly, I lay out the derivative of the utility with respect to $\pi_i(\boldsymbol{\alpha}_{t+k})$ as

$$\delta^{t+k}\Big(1 + \mathbb{1}\{\pi_i(\boldsymbol{\alpha}_{t+k}) < r_{i,t+k}\} \cdot \eta - \sum_{m=1}^{\infty} \delta^m \mathbb{1}\{\pi_i(\boldsymbol{\alpha}_{t+k+m}) < r_{i,t+k+m}\} \cdot \eta \cdot \frac{\partial r_{i,t+k+m}}{\partial \pi_i(\boldsymbol{\alpha}_{t+k})}\Big).$$

This derivative shows direct positive and indirect negative effects from a higher intrinsic payoff $\pi(\boldsymbol{\alpha}_{t+k})$ in addition to the standard positive effect (δ^{t+k}). First, the second term in the parentheses shows that it directly reduces the absolute value of the loss utility in the current period if the intrinsic payoff is in the loss region ($\pi_i(\boldsymbol{\alpha}_{t+k}) < r_{i,t+k}$). Second, the third term summarizes the indirect negative effect. It indirectly increases the absolute value of loss utilities in future periods when the intrinsic payoff is in the loss region ($\pi_i(\boldsymbol{\alpha}_{t+k+m}) < r_{i,t+k+m}$); a higher current payoff raises the reference point in the next period, the effect of which persists over the following periods ($\partial r_{i,t+k+m}/\partial \pi_i(\boldsymbol{\alpha}_{t+k}) > 0$). After all, the standard positive effect dominates the indirect negative effect, and Lemma 1 follows.

Lemma 1. *All the players choosing α^N every period, regardless of the history of cooperation or the initial reference point, $(s^N, s^N, \dots s^N)$, is a subgame-perfect equilibrium.*

Proof. See Appendix B. □

Given this result, I formally define a Nash-reversion strategy in Definition 4.

Definition 4. A Nash-reversion strategy with a cooperation path $\{\alpha_t^c\}_{t=1}^{\infty}$ for player i, $s(\{\alpha_t^c\}_{t=1}^{\infty})$, is a sequence of mapping $\{s_k(\{\alpha_t^c\}_{t=1}^{k})\}_{k=1}^{\infty}$ such that

$$s_k(\{\alpha_t^c\}_{t=1}^{k}) : A^{N(t-1)} \times \mathbb{R} \mapsto \begin{cases} \alpha_k^c & \text{if } \forall t \in \{1, 2, \dots, k-1\},\ \forall \text{j} \in \boldsymbol{N}, \alpha_{jt} = \alpha_t^c \ \text{ or if } k = 1 \quad \text{and} \\ \alpha^N & \text{otherwise.} \end{cases}$$

If someone deviates from a cooperation path $\{\alpha_t^c\}_{t=1}^{\infty}$, everyone starts implementing the Nash penalty with s^N from the next period

Another part that we must identify before examining cooperation paths is the best deviation against a Nash-reversion strategy. Given that the other players employ s^N from the next period, the best deviation maximizes the present value of utility. Similarly to the analysis of the Nash strategy, the marginal utility with respect to the own cooperation level has, possibly, two effects in addition to the standard positive effect on the intrinsic payoff in the current period: directly alleviating the current loss utility and indirectly aggravating the future loss utilities by raising future reference points. Nevertheless, the best deviation becomes the same as in the case without the reference dependence ($\eta = 0$) because the standard positive effect on the intrinsic payoff in the current period dominates the indirect effect of aggravating the future loss utilities. Formally, Lemma 2 states the result.

Lemma 2. *When the others play a Nash-reversion strategy $s(\{\alpha_t^c\}_{t=1}^{\infty})$, the best deviation in period t (α_t^b) is unique and the same as that without the reference dependence ($\eta = 0$). That is, for any $i \in \boldsymbol{N}$, $\forall r_{i,t} \in \mathbb{R}$,*

$$\underset{\alpha_{it}\in A}{\operatorname{argmax}}\left(\pi_i(\alpha_{it}, \boldsymbol{\alpha}_{-i,t}^c) + \ell(\alpha_{it}, \boldsymbol{\alpha}_{-i,t}^c | r_{i,t}) + \sum_{k=1}^{\infty} \delta^k \left[\tilde{\pi}(\alpha^N) + \ell(\boldsymbol{\alpha}^N | r_{i,t+k})\right]\right) = \underset{\alpha_{it}\in A}{\operatorname{argmax}}\, \pi_i(\alpha_{it}, \boldsymbol{\alpha}_{-i,t}^c)$$

where $\boldsymbol{\alpha}_{-i,t}^c$ is the vector of cooperation levels by all the players but i where $\alpha_{jt} = \alpha_t^c$ for all $j \neq i$.

Proof. See Appendix C. □

Having validated the penalty and identified the best deviation, I analyze the subgame-perfect equilibria with Nash-reverting strategies in the next section.

4 Subgame-Perfect Equilibria

This section studies the subgame-perfect equilibria with Nash-reverting strategies and elucidates how reference dependence affects cooperation levels. In the rest of the paper, I focus on cooperation paths that take values between the Nash cooperation and the first-best cooperation, $[\alpha^N, \bar{\alpha}]$. [7] Given this interval, the focus on the symmetric equilibria with Nash-reversion strategies, and Lemmas 1 and 2, I simplify notations in the following algebra using π_t and $b(\pi_t)$ where $\pi_t = \tilde{\pi}(\alpha_t)$, as if the players directly choose π_t instead of $\boldsymbol{\alpha}_t$, and a player earns $b(\pi_t)$ by the best deviation. The notation for a Nash-reversion strategy changes accordingly from $s(\{\alpha_t^c\}_{t=1}^{\infty})$ to $s(\{\pi_t^c\}_{t=1}^{\infty})$. Since $\tilde{\pi}$ is increasing on $[\alpha^N, \bar{\alpha}]$, I sometimes refer to the level of π_t as that of cooperation and, in particular, $\bar{\pi}$ as the first-best cooperation. A cooperation level greater than α^N and an intrinsic payoff greater than $\underline{\pi}$ are called *cooperative*. I assume the initial reference point falls into the same interval as intrinsic payoffs.

Assumption 3. *The initial reference point r_1 is not lower than the intrinsic payoff by the Nash cooperation and not higher than that by the first-best cooperation; that is, $\underline{\pi} \leq r_1 \leq \bar{\pi}$.*

In this game with the initial reference point r_1, $(s(\{\pi_t^c\}_{t=1}^{\infty}), s(\{\pi_t^c\}_{t=1}^{\infty}), \ldots, s(\{\pi_t^c\}_{t=1}^{\infty}))$ forms a subgame-perfect equilibrium if and only if, for all $t \in \mathbb{N}$, every player weakly prefers not deviating, $s(\{\pi_t^c\}_{t=1}^{\infty})$, to the best deviation. In the rest of this paper, I call a cooperation path $\{\pi_t^c\}_{t=1}^{\infty}$ *feasible* if and only if $(s(\{\pi_t^c\}_{t=1}^{\infty}), s(\{\pi_t^c\}_{t=1}^{\infty}), \ldots, s(\{\pi_t^c\}_{t=1}^{\infty}))$ forms a subgame-perfect equilibrium. Proposition 1 states the condition for a cooperation path $\{\pi_t^c\}_{t=1}^{\infty}$ to be feasible.

[7] This focus does not forbid a best deviation from taking a value lower than α^N.

Proposition 1. $\{\pi_t^c\}_{t=1}^{\infty}$ *is feasible if and only if,*

$$\begin{aligned}
\forall t \in \mathbb{N}, \quad & b(\pi_t^c) - \pi_t^c + \mathbb{1}\{r_t > \pi_t^c\} \cdot \eta \left[\min\{r_t, b(\pi_t^c)\} - \pi_t^c\right] \\
& \leq \sum_{k=1}^{\infty} \delta^k \Big[\pi_{t+k}^c - \mathbb{1}\{r_{t+k} > \pi_{t+k}^c\} \cdot \eta \left[r_{t+k} - \pi_{t+k}^c\right] \\
& \qquad - \left\{ \underline{\pi} - \mathbb{1}\{r_{t+k}^d > \underline{\pi}\} \cdot \eta \left[r_{t+k}^d - \underline{\pi} \right] \right\} \Big] \qquad (1) \\
& s.t. \forall k \in \mathbb{N}, \; r_{t+k} = \rho r_{t+k-1} + (1-\rho)\pi_{t+k}^c \\
& \forall k \in \mathbb{N}, \; r_{t+k}^d = \rho r_{t+k-1}^d + (1-\rho)\underline{\pi} \quad \textit{where } r_t^d = r_t
\end{aligned}$$

Proof. It follows from Lemmas 1 and 2. □

This condition inherits from a standard model a tension between the short–term gain and the long-term loss by a deviation. The left-hand side of the inequality expresses the short-term gain from a deviation in period t. It consists of improving the intrinsic payoff ($b(\pi_t^c) - \pi_t^c$) and, if any, reducing the loss utility. The right-hand side is the difference in the present value of utilities from the following periods between the equilibrium and the deviation paths, which I refer to as a long-term loss. The value of future utilities from the cooperative levels of cooperation is measured with reference points in the equilibrium path $\{r_{t+k}\}_{k=1}^{\infty}$, and that of a deviation is with reference points in the deviation path $\{r_{t+k}^d\}_{k=1}^{\infty}$.

To interpret the effects from reference dependence and those that are not, separately, suppose that there is no reference dependence ($\eta = 0$) and that the cooperation path is constant ($\pi_t^c = \pi^c$). Inequality (1) becomes

$$b(\pi^c) - \pi^c \leq \sum_{k=1}^{\infty} \delta^k \left[\pi^c - \underline{\pi}\right].$$

Then, the short-term gain on the left-hand side is convex and possibly becomes very large when the cooperative intrinsic payoff π^c is sufficiently high. On the other hand, the long-term loss on the right-hand side contains the standard loss that is linear in π^c. Therefore, when raising π^c, at some point, the standard long-term loss cannot keep up with the standard short-term gain, subject to the parameter values, and high cooperation cannot satisfy the inequality.

Reference dependence adds three effects to this tension. To see them clearly, I express the future reference points r_{t+k} and r^d_{t+k} in inequality (1) as the functions of the current reference point r_t and future intrinsic payoffs. With additional manipulations, inequality (1) becomes

$$\begin{aligned}
& b(\pi_t^c) - \pi_t^c + \mathbb{1}\{r_t > \pi_t^c\} \cdot \underbrace{\eta\left[\min\{r_t, b(\pi_t^c)\} - \pi_t^c\right]}_{\text{Loss-evading deviation incentive}} \\
& \leq \sum_{k=1}^{\infty} \delta^k \pi_{t+k}^c - \underbrace{\sum_{k=1}^{\infty} \mathbb{1}\{r_{t+k} > \pi_{t+k}^c\} \cdot \delta^k \eta \left[\rho^k r_t + \sum_{\ell=0}^{k-1} \rho^{k-1-\ell}(1-\rho)\pi_{t+\ell}^c - \pi_{t+k}^c\right]}_{\text{Diminishing future cooperation value}} \\
& \qquad - \sum_{k=1}^{\infty} \delta^k \underline{\pi} + \underbrace{\sum_{k=1}^{\infty} \delta^k \eta \left[\rho^k r_t + \rho^{k-1}(1-\rho) b(\pi_t^c) - \rho^{k-1}\underline{\pi}\right]}_{\text{Aversion to losing cooperation}}. \qquad (2)
\end{aligned}$$

First, this inequality shows that the long-term loss includes the aversion to losing cooperation. Notice that the last summation term on the right-hand side is non-negative and increases in the reference point r_t. After a deviation, a player incurs a loss utility every period. This loss utility is aggravated by a high reference point in the deviation period as because the reference point is persistent ($\rho > 0$). Intuitively, the more a player is used to a high payoff, the more she dislikes losing it by a deviation.

The second effect is a loss-evading deviation incentive that is operative only if a cooperative intrinsic payoff generates a loss, $r_t > \pi_t^c$. When the cooperative level of cooperation causes a loss, it incentivizes the players to avoid it by a deviation, captured by the third term on the left-hand side as an additional short-term gain. This term shows that given a cooperation level π_t^c, the magnitude of this effect weakly increases in the reference point r_t. The larger a loss is, the more room a player has to improve by a deviation. However, when a reference point becomes too high relative to π_t^c, the loss becomes too large, and it becomes impossible to completely avoid a loss by the best deviation ($r_t > b(\pi_t^c)$). In that case, the magnitude of the effect becomes unresponsive to further increase of a reference point.

The third effect diminishes the future cooperation value, which is existent only if a future cooperative intrinsic payoff π^c_{t+k} generates a loss, ($r_{t+k} > \pi^c_{t+k}$). This effect decreases the long-term loss by a deviation, captured by the second summation term on the right-hand side. Given a cooperation path $\{\pi^c_t\}_{t=1}^{\infty}$, this term increases in the reference point r_t; a reference point is persistent, and a higher reference point implies a more significant loss from a given π^c_{t+k}.

Whether a higher initial reference point r_t relaxes or tightens this inequality condition depends on whether future cooperative intrinsic payoffs generate losses against reference points. If there is no loss in any period of the equilibrium path, only the aversion to losing cooperation is operative; consequently, a higher r_t relaxes the condition. On the contrary, if there is a loss every period ($r_{t+k} > \pi^c_{t+k}$ for all k), all three effects are operative and strengthened by a higher initial reference point. The net effect is an additional incentive for a deviation; eq. (2) shows that the contributions of the initial reference point to the aversion to losing cooperation and the diminishing future cooperation value are both $\sum_{k=1}^{\infty} \delta^k \eta \rho^k r_t$ and cancel out each other. Thus, a higher initial reference point tightens the condition. I discuss the net effect in optimized paths in Section (5).

In the rest of the paper, I exclude an uninteresting case in which the first-best cooperation $\bar{\pi}$ is feasible from period 1, regardless of the initial reference point. Imposing Assumption 4 ensures it.

Assumption 4. *The first-best cooperation every period $\{\bar{\pi}\}_{t=1}^{\infty}$ cannot be feasible with a Nash-reversion strategy when the reference point is at the Nash cooperation level ($r_1 = \alpha^N$) as*

$$b(\bar{\pi}) - \bar{\pi} > \frac{\delta}{1-\delta}\left[\bar{\pi} - \underline{\pi}\right] + \frac{\delta\eta}{1-\delta\rho}\left[\rho\underline{\pi} + (1-\rho)b(\bar{\pi}) - \underline{\pi}\right].$$

The less patient (low δ) and the less loss-averse (low η) the players are, and the less persistent a reference point is (low ρ), the more likely Assumption 4 holds. The right-hand side of the inequality in Assumption 4 increases in δ; the magnitudes of the standard long-term loss and the aversion to losing cooperation are evaluated smaller with a lower discount factor. It also increases in ρ and η; the aversion to losing cooperation weakens when losses after a deviation are less persistent and when a player is less loss-averse. Another possible effect that supports the condition of Assumption 4 is a lucrative best deviation. The higher $b(\bar{\pi})$ is relative to $\bar{\pi}$ and $\bar{\pi}$, the more likely the inequality holds.

5 Gradual Development

In this section, I investigate a subgame-perfect equilibrium that maximizes the present value of utility among those feasible by a Nash-reversion strategy–optimal path. The result shows that cooperation develops gradually. The optimal path is the solution to the following problem, the constraints of which ensure that it is feasible with a Nash-reversion strategy following Proposition 1.

$$\begin{aligned} v^*(r_1) &= \max_{\{\pi_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \left[\pi_t - \mathbb{1}\{r_t > \pi_t\} \cdot \eta(r_t - \pi_t)\right] \\ s.t. \forall t \in \mathbb{N}\ &\pi_t \in \Gamma(r_t, v^*) \\ \forall t \in \mathbb{N}\ &r_{t+1} = \rho r_t + (1-\rho)\pi_t \end{aligned} \tag{3}$$

where the feasible set $\Gamma(r_t, v^*)$ is defined as

$$\begin{aligned} \Gamma(r_t, v^*) = \Bigg\{ \pi_t \in [\underline{\pi}, \bar{\pi}] : &\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi_t) - \mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta \left[r_t - b(\pi_t)\right] \\ &+ \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \leq v^*(r_t) \Bigg\} \end{aligned}$$

The constraint in the feasible set is simplified by summarizing the terms in the deviation path, and the inequality compares the value of the deviation path on the left-hand side and that of the equilibrium path on the right-hand side. To characterize the solution, I set up the corresponding functional equation for dynamic programming. It takes the reference point r_t as the state variable and the intrinsic payoff π_t as the control variable. I obtain Lemma 3.

Lemma 3. *The function v^* satisfies the following functional equation.*

$$v^*(r) = \max_{y \in \Gamma(r, v^*)} \left[y - \mathbb{1}\{r > y\} \cdot \eta(r - y) + \delta v^*(\rho r + [1 - \rho] y) \right] \tag{4}$$

Proof. See Appendix D. □

The functional equation (4) differs from a standard dynamic programming problem in that the correspondence $\Gamma(r, v^*)$ has the function v^* as an argument. Nevertheless, Lemma 3 states that the function v^* satisfies eq. (4), enabling us to characterize the value function v^* and the policy function $g^*(r) = \text{argmax}_{y \in \Gamma(r, v^*)}\, y - \mathbb{1}\{r > y\} \cdot \eta(r - y) + \delta v^*(\rho r + [1 - \rho] y)$.

The properties of the value function v^* and the policy function g^* depend on whether the first-best cooperation $\bar{\pi}$ is sustainable. I define sustainability in Definition 5.

Definition 5. A cooperation level π (or a reference point r) is called sustainable if the constant cooperation path at that level $\{\pi\}_{t=1}^{\infty}$ (or $\{r\}_{t=1}^{\infty}$) is feasible with a Nash-reversion strategy when the initial reference point is π (or r). Otherwise, it is called unsustainable.

The constant cooperation path at the level of the initial reference point does not change the reference point over periods or generate a loss. The condition for a cooperation π to be feasible follows from eq. 1 as

$$b(\pi) - \pi \leq \frac{\delta}{1 - \delta} \left[\pi - \underline{\pi} \right] + \frac{\delta \eta}{1 - \delta \rho} \left[\rho \pi + (1 - \rho) b(\pi) - \underline{\pi} \right]. \tag{5}$$

Given this definition, I partition all the cases into two.

Case 1. The first-best cooperation is sustainable.

Case 2. The first-best cooperation is unsustainable.

In the same way as the condition of Assumption 4, given $\pi = \bar{\pi}$, the right-hand side of the inequality 5 increases in δ, η, and ρ; the more patient (higher δ) and the more loss-averse (higher η) the players are, the more likely it falls into Case 1. Also, the more persistent a reference point is (higher ρ), the more likely it is Case 1. These increase the magnitude of the long-term loss. On the other hand, a more lucrative best deviation (higher $b(\bar{\pi})$) makes Case 2 more likely by raising the short-term gain. In the rest of this section, I first provide analytical results for Case 1. Then, I provide analytical results for Case 2. Finally, I provide numerical examples of Case 2 to enrich the analysis.

5.1 Case 1: Eventually Reaching First-Best Cooperation

In Case 1, the main properties of the policy function g^* are summarized in Proposition 2.

Proposition 2 (Case 1)**.** *The policy function* g^* is continuous on $[\underline{\pi}, \bar{\pi}]$ and strictly increasing and strictly concave on $[\underline{\pi}, \gamma]$. Also, $r < g^*(r) < \bar{\pi}$ *on* $[\underline{\pi}, \gamma)$*, and* $g^*(r) = \bar{\pi}$ *on* $[\gamma, \bar{\pi}]$. Consequently, $\bar{\pi}$ is the unique steady state.

Proof. See Appendix E. □

According to these properties, Figure 1 illustrates the optimized cooperation path for the players with the initial reference point at $\underline{\pi}$. The intrinsic payoff π_t (blue dots) is located by $g^*(r_t)$, and then the reference point in the next period r_{t+1} (red dots) is determined as the weighted average of π_t and r_t. The optimized cooperation $g^*(r_t)$ is always higher than r_t (except for $r = \bar{\pi}$), which raises the reference point in the next period, and, consequently, the optimized cooperation in the next period. This way, the cooperation level gradually and monotonically rises and reaches the first-best level $\bar{\pi}$ when the reference point reaches or exceeds γ.

What generates the monotonically increasing cooperation is the monotonically strengthening aversion to losing cooperation ($\delta\eta\rho r_t/(1-\delta\rho)$ in $\Gamma(r_t, v^*)$). Experiencing a higher utility raises the reference point and aggravates the loss utilities in the deviation path. Consequently, the aversion strengthens and enables the players to increase the cooperation levels in the next period, raising their reference points further. The higher reference point further aggravates the loss utilities in the deviation path and makes a further higher cooperation level feasible. The repetition of this process continues until the reference point reaches the upper limit $\bar{\pi}$. This cooperation path never generates a loss; therefore, it is the only operative effect of reference dependence.

5.2 Case 2: Not Reaching First-Best Cooperation

In Case 2, the optimized cooperation path is affected additionally by the other forces from reference dependence; there is an interval on the reference point r where an intrinsic utility as high as the reference point ($y > r$) is not feasible.

Proposition 3 (Case 2)**.** *The policy function g^** is continuous, strictly increasing, and strictly concave on $[\underline{\pi}, \bar{z}]$ where $\bar{z}$ is the highest sustainable cooperation level that is implicitly defined by

$$b(\bar{z}) - \bar{z} = \frac{\delta}{1-\delta}\left[\bar{z} - \underline{\pi}\right] + \frac{\delta\eta}{1-\delta\rho}\left[\rho\bar{z} + (1-\rho)b(\bar{z}) - \underline{\pi}\right].$$

Also, $r < g^*(r) < \bar{z}$ for $r < \bar{z}$, $g^*(\bar{z}) = \bar{z}$, and *$g^*(r) < \bar{z}$ for $r > \bar{z}$.* Consequently, $\bar{z}$ is the unique steady state.

Proof. See Appendix F. □

When the players' initial reference point r_1 is not as high as $\bar{z}$, which we expect for new relationships, the development of cooperation exhibits gradualism. The properties of the policy function on $[\underline{\pi}, \bar{z}]$ are the same as those of $[\underline{\pi}, \gamma]$ in Case 1. Roughly speaking, this result is because the optimized paths of the payoff $\{\pi\}_{t\in 1}^{\infty}$ from any reference point $r_1 \in [\underline{\pi}, \gamma]$ do not contain $\pi_t < r_t$. Therefore, it never generates a loss utility, and only the aversion to losing cooperation is operative in the same way as Case 1. On the other hand, when a reference point is unsustainable ($r > \bar{z}$), $g^*(r)$ is cannot exceed $\bar{z}$, let alone r. Consequently, the highest sustainable cooperation $\bar{z}$ is the unique steady state of this model in Case 2. The result of $g^*(r) < \bar{z}$ for $r \in (z, \bar{\pi}]$ is an essential property in analyzing the response of cooperation to a change in the economic environment in Section(6).

The cooperation level not exceeding $\bar{z}$ with a reference point higher than $\bar{z}$ reflects that the net marginal effect from a higher reference point is an additional deviation incentive. When a reference point is above $\bar{z}$, the standard gain by a deviation becomes so large that the players cannot set $\pi_t \geq r_t$. Consequently, they have to follow the equilibrium path with a loss in the current period and either losses or lower intrinsic payoffs in future periods.[8] The latter, the diminishing future cooperation value, cancels out the aversion to losing cooperation. Whether or not a player deviates, she incurs losses or something equivalent in future. After all, the remaining effect is the loss-evading deviation incentive, which disables the players to keep the cooperation level even at $\bar{z}$.

A higher reference point inhibits cooperation more severely, which I can prove for a case with Assumption 5 as Proposition 4.

Assumption 5. *The highest sustainable $\bar{z}$ is sufficiently close to $\bar{\pi}$.*

Proposition 4 (Case 2 with Additional Assumption)**.** *Given Assumption 5, g^* is strictly decreasing on* $[\bar{z}, \bar{\pi}]$.

Proof. See Appendix F. □

The decreasing g^* reflects that the magnitude of the loss-evading deviation incentive increases with a reference point r. In general, it increases with r up to the point where the best deviation payoff reaches a reference point ($b(g(r)) = r$), which makes it infeasible to obtain analytical results beyond $g(r) < \bar{z}$. Assumption 5 ensures that the best deviation payoff always exceeds a reference point ($b(g(r)) > r$ for all r), and provides the additional analytical result.

5.3 Numerical Examples of Case 2

This subsection provides numeric examples of Case 2 to obtain more sense of the optimized paths. Figure 2 shows one with high ρ and δ that is not very low. The policy function $g^*(r)$ (solid blue) is

[8]When a reference point becomes below $\bar{z}$ in period $t+1$ reflecting π_t lower than $\bar{z}$, there is no loss onward. Instead, the players receive intrinsic payoffs lower than $\bar{z}$.

upward-sloping until it intersects the 45-degree (dashed green) line and then becomes downward-sloping. The intersection corresponds to the highest sustainable cooperation $\bar{z}$, and the properties of g^* are exactly as stated in Propositions 3 and 4. The other solid line (orange) locates the reference point in the next period ($\rho r + (1 - \rho)g^*(r)$), which increases even after the intersection. In this example, when the initial reference point is unsustainable ($r > \bar{z}$), the reference point gradually declines, the cooperation gradually rises, and both converge to the intersection from the right in the graph. Throughout this converging path, the players incur loss utilities.

An unsustainable reference point inhibits cooperation more severely when the players are less patient and when a reference point is less persistent. Figure 3 shows another example of Case 2 in which the discount factor δ and the persistence of a reference point ρ are smaller than Example 1 in Figure 2. There are four differences from Figure 2. First, the optimized cooperation g^* slope in the sustainable region ($\underline{\pi} \leq r \leq \bar{z}$) flattens, reflecting that the aversion to losing cooperation becomes weaker by less persistent losses in the deviation path and more discounting of given future losses. Second, the highest sustainable cooperation $\bar{z}$ becomes smaller, caused by the smaller standard long-term loss with more discounting and the weakening aversion to losing cooperation. Third, the downward slope in the unsustainable region ($\bar{z} < r$) close to $\bar{z}$ becomes steeper. This steeper slope reflects that the magnitude of the loss-evading deviation incentive becomes relatively greater than in Case 1. A smaller discount factor and a less persistent reference point reduce the long-term losses, whereas the standard short-term gain or the loss-evading deviation incentive is unaffected. Consequently, a given increase in the loss-evading deviation incentive tightens the constraint in $\Gamma(r, v^*)$ more. Finally, there are kinks in $g^*(r)$ and, subsequently, $\rho r + (1 - \rho)g^*(r)$ around $r = 0.95$. Where the reference point is higher than this point, the best deviation cannot eliminate a loss in the current period ($r > b(g^*(r))$. A higher reference point in this region does not generate an additional incentive for a deviation.

6 Rough Transition

In this section, I study the model's implications for transitions of cooperation levels when the economic environment changes. To this end, I consider Case 2 for richer implications and assume that a reference point is at the steady state $\bar{z}$, corresponding to a mature relationship. As a similar exercise, I provide Appendix H that discusses how a cooperation level fluctuates and moves back to a steady state when a transitory shock shifts a reference point away from the steady state.

Reference dependence generates asymmetric transitional dynamics when the steady state $\bar{z}$, the highest sustainable cooperation, shifts. The steady state $\bar{z}$ shifts when any of function b and parameters $(\underline{\pi}, \bar{\pi}, \delta, \eta, \rho)$ change, which happens for various reasons; for example, an entrant or an exit in a Cournot competition changes b, $\underline{\pi}$, and $\bar{\pi}$. [9]In the following analysis, we consider a Cournot competition in which the incumbents are tacitly colluding and achieving the highest sustainable profit $\bar{z}$ below the monopolist level ($\bar{\pi}$).

A cooperation path in transitional dynamics is monotonic when the steady state rises. Suppose an incumbent exits the market at the end of period t. Then, it increases the profits of the remaining incumbents from period $t+1$, which is a positive shock for the remaining firms. Consequently, $\bar{z}$ rises and becomes higher than the reference point r_{t+1} because r_{t+1} is not directly affected.[10] Given that $g(r) > r$ when $r < \bar{z}$, the remaining firms can increase their profits in period $t+1$. In the following periods, the profits monotonically rise to $\bar{z}$. Figure 4 shows this path by a solid blue line. It uses the same parameters except for the number of firms as in Example 2 in Section 5. Figure 3 of Example 2 uses a functional form of Cournot competition among 15 firms. In Figure 4, the number of firms decreases from 15 to 14 in period 2.

In contrast, a negative shock causes an excessive decline in the profits in the short term, like the solid orange line in Figure 4 where the number of firms increases to 16 in Example 2. When

[9]A more straightforward example of the shift of $\bar{z}$ is a permanent intrinsic payoff shock. That is, $\pi^{new}_{t+k}(\boldsymbol{\alpha}) = \pi^{old}_{t+k}(\boldsymbol{\alpha}) + \epsilon$ for all $\boldsymbol{\alpha} \in A^N$ and all $k \in \mathbb{N}$ for a shock at the beginning of period $t+1$. This shock shifts everything but $\bar{z}$ to the same extent and does not change the constraint in $\Gamma(r, v^*)$ or the shapes of g^*.

[10]Also, $\bar{\pi}$ and $\underline{\pi}$ rise, and $b(\pi)$ changes.

a new firm enters the market and joins the tacit collusion, individual firms' profits fall, which is a negative shock. Thus, $\bar{z}$ declines.[11] Given that $g(r) < \bar{z}$ for $r > \bar{z}$, the incumbents' profits in period $t + 1$ (period 2 in Figure 4) become lower than the profits in the new steady state $\bar{z}$, let alone the previous steady-state level. This overshoot reflects the loss-evading deviation incentive as discussed in Section 5. The firms are used to a high level of profit. Even after it becomes infeasible with tacit collusion by an entrant, they try to cling to it by deviating from the collusion. This incentive makes their cooperation difficult and results in a low profit. After the drop in period $t + 1$, the profit converges over periods to the new steady state level by taking time for the firms to become used to a lower profit level.

7 Conclusion

This paper provides a new explanation of the gradual development of cooperation. In models of gradualism with myopic players and incomplete information, the existence of myopic players forces players to start cooperation at a low level. In comparison, the reference dependence raises the initial level of cooperation and facilitates it further over time. Thus, cooperation develops as if the players are "starting small." A limitation of the analysis is that I analyze subgame perfect equilibria by Nash-reversion strategies to obtain the implications of reference dependence in a simple way. The investigation of the optimal punishment in the presence of reference dependence potentially provides additional results.

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[11] I assume the entrant has the same reference point as the incumbents when it enters.

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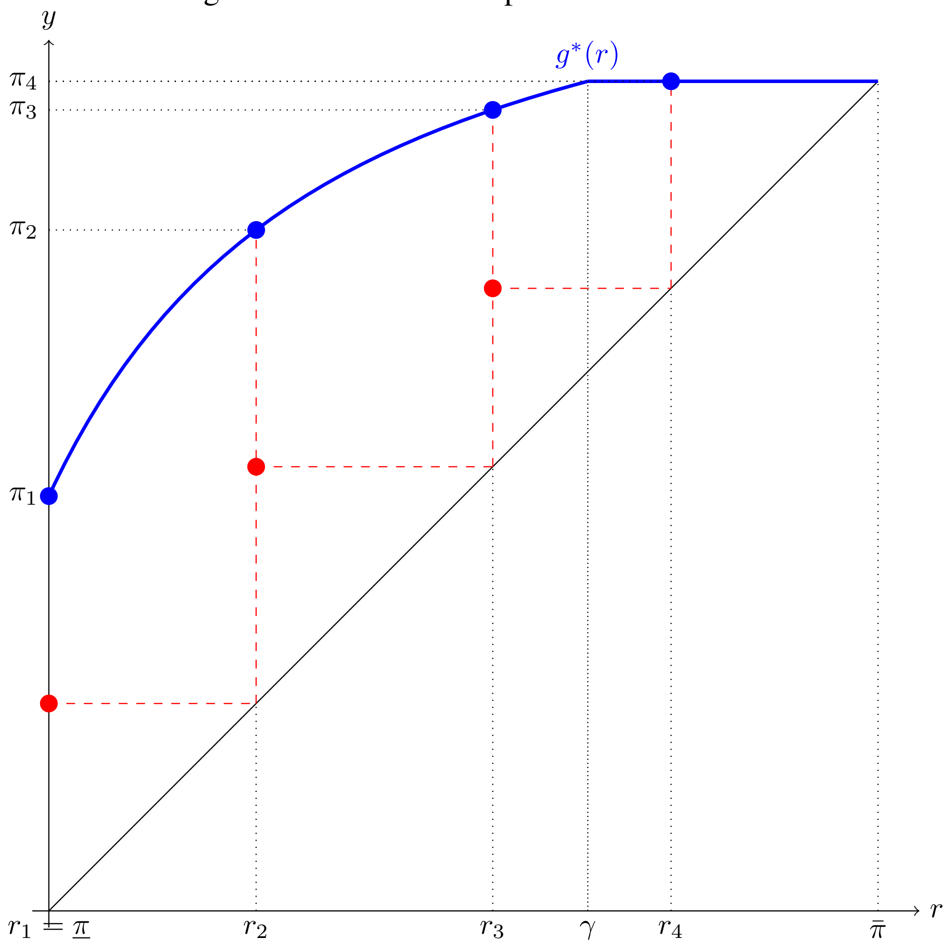
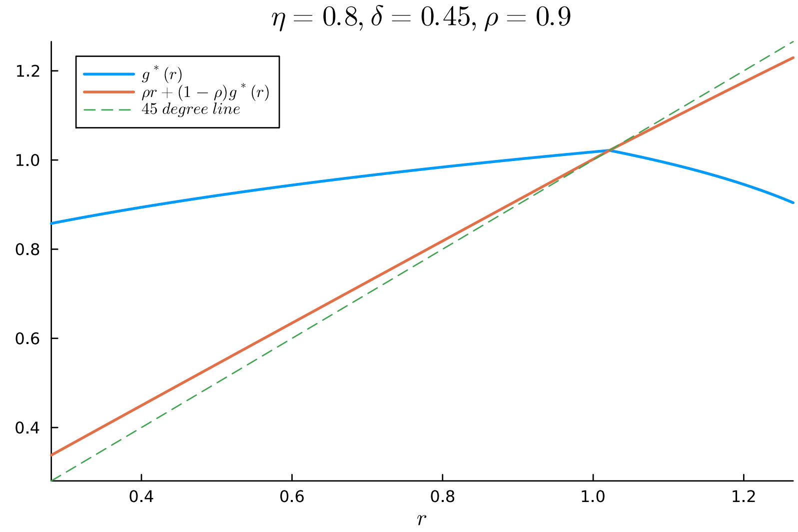
Figures

Figure 1: Gradual Development Path in Case 1



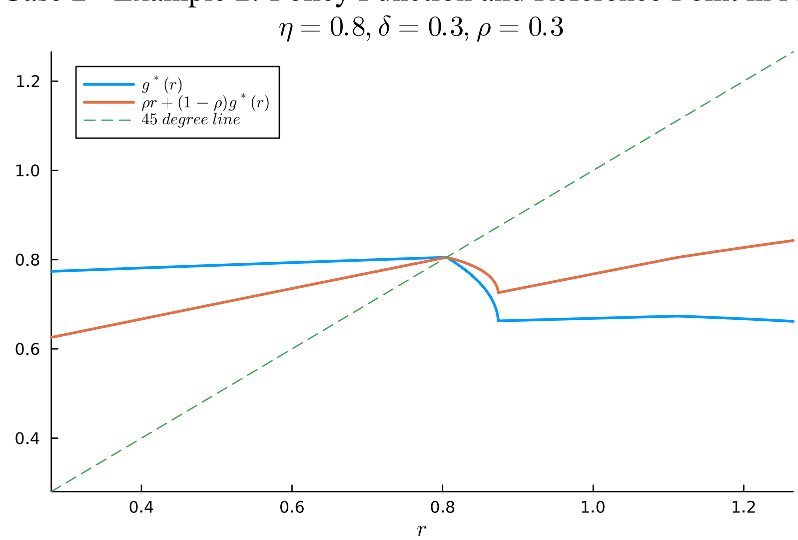
Notes: The solid blue line depicts the policy function g^* according to the properties in Proposition 2 for Case 1. The blue dots locate $\{\pi_1, \pi_2, \pi_3, \pi_4\}$ and the red dots locate $\{r_2, r_3, r_4\}$ of the optimized cooperation path with $r_1 = \underline{\pi}$.

Figure 2: Case 2 - Example 1: Policy Function and Reference Point in Next Period



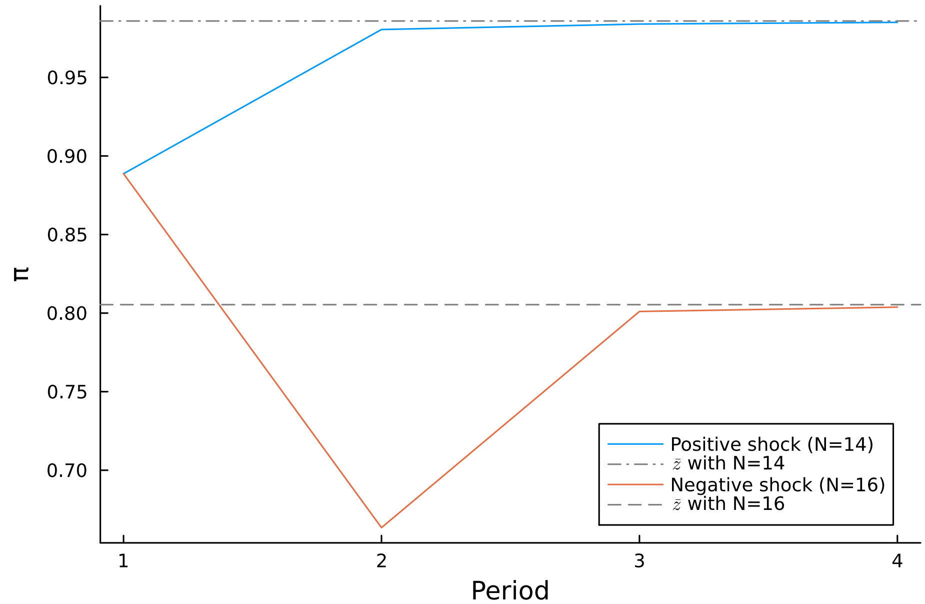
Notes: The policy function g^* is obtained for the case where $\eta = 0.8$, $\delta = 0.45$, and $\rho = 0.9$. The intrinsic payoff $\pi(\boldsymbol{\alpha})$ is derived from Cournot competition by $N = 15$ firms with common marginal cost $c = 1$ with the inverse demand function $10 - \sum x_i$ where x_i is the quantity supplied by firm i.

Figure 3: Case 2 - Example 2: Policy Function and Reference Point in Next Period



Notes: The policy function g^* is obtained for the case where $\eta = 0.8$, $\delta = 0.3$, and $\rho = 0.3$. The intrinsic payoff $\pi(\boldsymbol{\alpha})$ is derived from Cournot competition by $N = 15$ firms with common marginal cost $c = 1$ with the inverse demand function $10 - \sum x_i$ where x_i is the quantity supplied by firm i.

Figure 4: Transitional Dynamics from $\bar{z}$ with $N = 15$ in Cournot competition



Notes: These lines are drawn based on the policy function g^* with $\eta = 0.8$, $\delta = 0.3$, and $\rho = 0.3$ for different number of firms N in collusion under Cournot competition from which the intrinsic payoff $\pi(\alpha)$ is derived. Firms have a common marginal cost $c = 1$ with the inverse demand function $10 - \sum x_i$ where x_i is quantity supplied by firm i. In period 1, $r_1 = \pi_1 = \bar{z}$ for $N = 15$. In period 2 and later, π_t moves according to the policy functions for $N = 14$ (blue) and $N = 16$ (orange) with $r_2 = \pi_1$.

Appendix

A Microfoundation

I provide two microfoundations of the intrinsic payoff function π_i by showing the payoff functions in these two economic problems satisfy Assumptions 1 and 2.

A.1 Collusion in Cournot Competition

The first one is collusion in Cournot competition. N firms with a common marginal cost of c face the inverse demand function $a - \beta(\sum x_i)$ where x_i is quantity supplied by firm i, $(a-c)/2N\beta \leq x_i \leq (a-c)/N\beta$, and $a > c$. The lower bound of x_i is set such that the aggregate supply achieves the monopolist's profit. The upper bound is set such that the price becomes non-negative. The profit for firm i is

$$\pi_i = \left(a - \beta \sum_{j \neq i} y_j - \beta x_i \right) x_i - c x_i.$$

Let $\alpha_i = (a-c)/N\beta - x_i$. Then, $\bar{\alpha} = (a-c)/2N\beta$ and

$$\frac{d\tilde{\pi}(\alpha)}{d\alpha} = -\frac{d\left[(a - \beta N x)\, x - cx\right]}{dx} = c - a + 2\beta N x \begin{cases} > 0 & \alpha < \bar{\alpha} \\ = 0 & \alpha = \bar{\alpha} \end{cases}$$

Thus, Assumption 1 is satisfied. The best-response payoff is calculated as

$$b(\pi) = \frac{1}{4b} \left[(a-c) \cdot \frac{(N+1) - (N-1)\sqrt{1 - \frac{4N\beta\pi}{(a-c)^2}}}{2N} \right]^2 .$$

This b is convex, and $\pi = (a-c)^2/(N+1)^2 b$ is the unique solution to $b(\pi) = \pi$ as Assumption 2.

A.2 Trade Liberalization

The second one is trade liberalization. I consider a partial equilibrium model of trade between two symmetric countries, home and foreign. It is a special case of the model of Bond and Park (2002) that can have asymmetric sizes. I denote variables in foreign by those with superscript $*$. There are two goods, 1 and 2. The demand function for good k is $D_k(p_k) = A - Bp_k$ in both countries. The supply function for good k is given by $X_k(p_k) = \alpha_k + \beta p_k$ and $X_k^*(p_k^*) = \alpha_k^* + \beta^* p_k^*$ where $\beta = \beta^*$. By assuming $\alpha_1 - \alpha_1^* = \alpha_2^* - \alpha_2 > 0$ and $\alpha_1 = \alpha_2^*$, home (foreign) exports good 1 (2) and imports good 2 (1). Each country imposes specific tariff $t \in [0, T]$ on its import where $T = (\alpha_2^* - \alpha_2)/(B + \beta)$ is the lowest prohibitive tariff in this model. As a result, $p_1^* = p_1 + t^*$ and $p_2 = p_2^* + t$. Let p_k and p_k^* be the prices of good $k \in \{1, 2\}$. Then, the welfare is derived as the sum of consumer surplus, producer surplus, and tariff revenue. It is given by

$$W(t, t^*) = \sum_{k=1,2} \int_{p_k}^{A/B} D_k(u)du + \sum_{k=1,2} \int_{-\alpha/\beta}^{p_k} X_k(u)du + t(D_m(p_m) - X_m(p_m))$$

where m is 2 for home and 1 for foreign. Take derivatives with respect to t and t^*.

$$\begin{aligned}
\frac{\partial W(t, t^*)}{\partial t} &= M\left(1 - \frac{\partial p_2}{\partial t}\right) + t\frac{\partial M}{\partial p_2}\frac{\partial p_2}{\partial t} = \frac{\alpha_2^* - \alpha_2}{4} - \frac{B + \beta}{2}t \\
\frac{\partial W(t, t^*)}{\partial t^*} &= -M\frac{\partial p_1}{\partial t^*} = -\frac{\alpha_2^* - \alpha_2}{4} + \frac{B + \beta}{4}t^*
\end{aligned}$$

where $M = (\alpha_2^* - \alpha_2)/2 - (B + \beta)t/2$ is the net import of home, and the second equality in each row follows from the equilibrium good price. From these derivatives, the welfare can be derived as

$$W(t, t^*) = \frac{\alpha_2^* - \alpha_2}{4}(t - t^*) - (B + \beta)\left(\frac{t^2}{4} - \frac{t^{*2}}{8}\right) + C$$

where C is constant. Let $\alpha = T - t$, $\bar{\alpha} = T$, and $\tilde{\pi}(\alpha) = W(T - \alpha, T - \alpha)$. Then, Assumption 1 is satisfied as:

$$\frac{d\tilde{\pi}(\alpha)}{d\alpha} = \frac{B+\beta}{4} t \begin{cases} = 0 & \alpha = \bar{\alpha} \\ > 0 & \alpha < \bar{\alpha} \end{cases}$$

The best-response payoff is calculated as:

$$b(\pi) = \frac{(\alpha_2^* - \alpha_2)^2}{16(B+\beta)} - \frac{\alpha_2^* - \alpha_2}{4} 2 \sqrt{\frac{2(C - \pi)}{B+\beta}} + 2C - \pi$$

This function b is convex, $\pi = -(\alpha_2^* - \alpha_2)^2/32(B+\beta) + C$ is the unique solution to $b(\pi) = \pi$ as Assumption 2.

B Proof of Proposition 1

Consider a subgame starting from period t with the reference points $(r_{1,t}, r_{2,t})$. The utility for player i is given by $U_{i,t}(\{\boldsymbol{\alpha}_{t+k}\}_{k=0}^{\infty}, r_{i,t})$. The derivative of the utility with respect to $\alpha_{i,t+m}$ is:

$$\frac{\partial U_{i,t}}{\partial \alpha_{i,t+m}} = \delta^m \frac{\partial \pi_i(\alpha_{i,t+m}, \boldsymbol{\alpha}_{-i,t+m}}{\partial \alpha_{i,t+m}} \Phi$$

where

$$\Phi = 1 + \mathbb{1}\{\pi(\alpha_{i,t+m}, \boldsymbol{\alpha}_{-i,t+m}) < r_{i,t+m}\}\eta - \sum_{n=1}^{\infty} \delta^n \mathbb{1}\{\pi(\alpha_{i,t+m+n}, \boldsymbol{\alpha}_{-i,t+m}) < r_{i,t+m+n}\} \cdot \eta \cdot \frac{\partial r_{i,t+m+n}}{\partial \pi(\alpha_{i,t+m}, \boldsymbol{\alpha}_{-i,t+m})}$$

Φ is positive as:

$$\Phi \geq 1 - \sum_{n=1}^{\infty} \delta^n \eta \cdot \frac{\partial r_{i,t+m+n}}{\partial \pi_i(\alpha_{i,t+m}, \boldsymbol{\alpha}_{-i,t+m})} = 1 - \eta \sum_{n=1}^{\infty} \delta^n \cdot (1-\rho)\rho^{n-1} = 1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho} > 0$$

where the last inequality follows from $\eta \in (0,1]$ and $\delta, \rho \in (0,1)$. Thus, $\partial U_{i,t}/\partial \alpha_{i,t+m}$ and $\partial \pi_i(\alpha_{i,t+m}, \boldsymbol{\alpha}_{-i,t+m})/\partial \alpha_i$ have the same sign for all $r_{i,t} \in \mathbb{R}$, for all $\{\boldsymbol{\alpha}_{t+k-1}\}_{k=1}^{m} \in A^m$, and for all $\{\boldsymbol{\alpha}_{t+k-1}\}_{k=m+2}^{\infty} \in A^{\infty}$. This result implies

$$\operatorname*{argmax}_{\alpha_{i,t+m} \in A} U_{i,t}(\{\alpha_{i,t+k}, \boldsymbol{\alpha}_{-i,t+k}\}_{k=1}^{\infty}, r_{i,t}) = \operatorname*{argmax}_{\alpha_{i,t+m} \in A} \pi_i(\alpha_{i,t+m}, \boldsymbol{\alpha}_{-i,t+m})$$

for all $m \in \mathbb{N}$. Then, it follows that $(s^N, s^N, \ldots, s^N)$ is a Nash equilibrium in this subgame. Furthermore, given that this result holds for all $r_{i,t} \in \mathbb{R}$, it is subgame-perfect .

C Proof of Lemma 2

Suppose player i deviates in period t when the others play $s(\{\alpha^c_{t+k}\}_{k=0}^{\infty}, r_{i,t})$. Then, the derivative of utility with respect to $\alpha_{i,t}$ is given by

$$\begin{aligned}
&\frac{\partial \pi_i(\alpha_{i,t}, \boldsymbol{\alpha}^c_{-i,t})}{\partial \alpha_{i,t}} \left(1 + \mathbb{1}\{\pi_i(\alpha_{i,t}, \boldsymbol{\alpha}^c_{-i,t}) < r_t\} \cdot \eta - \sum_{k=1}^{\infty} \delta^k \cdot \mathbb{1}\{\tilde{\pi}(\alpha^N) < r_{i,t+k}\} \cdot \eta \cdot \frac{\partial r_{i,t+k}}{\partial \pi_i(\boldsymbol{\alpha}_t)}_{|\boldsymbol{\alpha}_t = (\alpha_{i,t}, \boldsymbol{\alpha}^c_{-i,t})} \right) \\
=&\frac{\partial \pi_i(\alpha_{i,t}, \alpha^c_t)}{\partial \alpha_{i,t}} \Phi'
\end{aligned}$$

where $\Phi' = 1 + \mathbb{1}\{\pi_i(\alpha_{i,t}, \boldsymbol{\alpha}^c_{-i,t}) < r_t\} \cdot \eta - \sum_{k=1}^{\infty} \delta^k \cdot \mathbb{1}\{\tilde{\pi}(\alpha^N) < r_{i,t+k}\} \cdot \eta \cdot (\partial r_{i,t+k}/\partial \pi_i(\boldsymbol{\alpha}_t))$ evaluated at $\boldsymbol{\alpha}_t = (\alpha_{i,t}, \boldsymbol{\alpha}^c_{-i,t})$. Then,

$$\Phi' \geq 1 - \eta \sum_{k=1}^{\infty} \delta^k \frac{\partial r_{i,t+k}}{\partial \pi_i(\boldsymbol{\alpha}_t)}_{|\boldsymbol{\alpha}_t = (\alpha_{i,t}, \boldsymbol{\alpha}^c_{-i,t})} = 1 - \eta \sum_{k=1}^{\infty} \delta^k \cdot (1-\rho)\rho^{k-1} = 1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho} > 0$$

where the last inequality follows from $\eta \in (0,1]$ and $\delta, \rho \in (0,1)$. Thus, the sign of $(\partial \pi_i(\alpha_{i,t}, \alpha^c_t)/\partial \alpha_{i,t})\Phi'$ equals to that of $\partial \pi_i(\alpha_{i,t}, \alpha^c_t)/\partial \alpha_{i,t}$, which implies the lemma.

D Proof of Lemma 3

I show the function v^* satisfies (4). v^* is bounded as: $\underline{\pi}/(1-\delta) \leq v^* \leq \bar{\pi}/(1-\delta)$. Since $\pi_t = \underline{\pi}$ is always feasible, $\Gamma(r_t, v^*)$ is nonempty for all r_t. Let $\Psi(r_t) \equiv \left\{\{\pi_k\}_{k=t}^{\infty} \in [\underline{\pi}, \bar{\pi}]^{\infty} : \pi_k \in \Gamma(r_k, v^*),\ s.t.\ r_{k+1} = \rho r_k + (1-\rho)\pi_k\right\}$ be the set of feasible cooperation paths from r_t. Given $\rho r_t + (1-\rho)\pi_t^o \in [\underline{\pi},\bar{\pi}]$, there exists an optimized path $\{\pi_k\}_{k=t+1}^{\infty}$ such that $v^*(\rho r_t + [1-\rho]\pi_t^o) = F(\{\pi_k\}_{k=t+1}^{\infty})$ where $F(\{\pi_k\}_{k=t+1}^{\infty}) = \sum_{k=t+1}^{\infty} \delta^{k-t}\Big[\pi_k + \mathbb{1}\{r_k > \pi_k\} \cdot \eta(r_k - \pi_k)\Big]$. It follows that, given $\pi_t, r_t \in [\underline{\pi},\bar{\pi}]$,

$$
\begin{aligned}
&\pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o) + \delta v^*(\rho r_t + [1-\rho]\pi_t^o) \\
&= \pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o) + \delta \max_{\{\pi_k\}_{k=t+1}^{\infty} \in \Psi([1-\rho]r_t + \rho\pi_t^o)} F(\{\pi_k\}_{k=t+1}^{\infty}) \\
&\leq \max_{\{\pi_k\}_{k=t}^{\infty} \in \Psi(r_t)} \{\pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o) + \delta F(\{\pi_k\}_{k=t+1}^{\infty})\} = v^*(r_t)
\end{aligned} \tag{6}
$$

where I use the fact that $\{\pi_t^o, \{\pi_k\}_{k=t+1}^{\infty}\} \in \Psi(r_t)$ when $\{\pi_k\}_{k=t+1}^{\infty} \in \Psi([1-\rho]r_t + \rho\pi_t^o)$. At the same time, given $r_t \in [\underline{\pi},\bar{\pi}]$,

$$
\begin{aligned}
v^*(r_t) &= \max_{\{\pi_k\}_{k=t}^{\infty} \in \Psi(r_t)} F(\{\pi_k\}_{k=t}^{\infty}) = \pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o) + \delta F(\{\pi_k^o\}_{k=t+1}^{\infty}) \\
&\leq \pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o) + \delta v^*(\rho r_t + [1-\rho]\pi_t^o)
\end{aligned} \tag{7}
$$

where $\{\pi_t^o, \{\pi_k^o\}_{k=t+1}^{\infty}\} = \text{argmax}_{\{\pi_k\}_{k=t}^{\infty} \in \Psi(r_t)} F(\{\pi_k\}_{k=t}^{\infty})$. Thus, it follows from inequalities (6) and (7) that

$$
v^*(r_t) = \pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o) + \delta v^*(\rho r_t + [1-\rho]\pi_t^o)
$$

v^* satisfies the functional equation (4).

E Proof of Proposition 2

I first obtain Lemma 4 that enables us to exclude the possibility of $v^*(\underline{\pi}) = \underline{\pi}/(1-\delta)$.

Lemma 4. *There exists $\pi' > \underline{\pi}$ such that $v^*(\underline{\pi}) \geq \pi'/(1-\delta)$.*

Proof. Consider a constant cooperation path $\{\pi\}_{t=1}^{\infty}$ in problem 3. Given $r_1 = \underline{\pi}$, the following inequality is sufficient for $v^*(\underline{\pi}) \geq \pi/(1-\pi)$.

$$\forall t,\ \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \leq \frac{\pi}{1-\delta} \tag{8}$$

which does not have indicator functions because $\pi \geq r_t$ for all t. Then, inequality (8) for $t = 1$ becomes

$$f(\pi) \equiv \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi) - \frac{\pi}{1-\delta} + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\underline{\pi} \leq 0$$

Since $b(\underline{\pi}) = \underline{\pi}$, $f(\underline{\pi}) = 0$. Take derivative of $f(\pi)$,

$$\begin{aligned} f'(\pi) &= \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b'(\pi) - \frac{1}{1-\delta} \\ &= \left(1 - \frac{\delta\eta}{1-\delta\rho}\right)\left(\sum_{j\neq i} \frac{\partial \pi_i(\alpha_i, \boldsymbol{\alpha}^c{}_{-i})}{\partial \alpha_{j,t}}_{|\alpha_i = \operatorname{argmax} \pi_i(\alpha_i, \boldsymbol{\alpha}^c{}_{-i})}\right) \frac{d\alpha_t}{d\tilde{\pi}(\alpha_t)}_{|\alpha_t = \alpha^c} - \frac{1}{1-\delta} \end{aligned}$$

where the second equality follows from the first order condition for $\alpha_i = \operatorname{argmax} \pi_i(\alpha_i, \boldsymbol{\alpha}^c{}_{-i})$. Evaluate this at $\pi = \underline{\pi}(\boldsymbol{\alpha}^c = \boldsymbol{\alpha}^N)$,

$$\begin{aligned} f'(\underline{\pi}) &= \left(1 - \frac{\delta\eta}{1-\delta\rho}\right)\left(\sum_{j\neq i} \frac{\partial \pi_i(\boldsymbol{\alpha}^N)}{\partial \alpha_{j,t}}\right) \frac{d\alpha_t}{d\tilde{\pi}(\alpha_t)}_{|\alpha_t = \alpha^c} - \frac{1}{1-\delta} \\ &= \left(1 - \frac{\delta\eta}{1-\delta\rho}\right)\left(\sum_{j\neq i} \frac{\partial \pi_i(\boldsymbol{\alpha}^N)}{\partial \alpha_{j,t}}\right)\left(\sum_{j\neq i} \frac{\partial \pi_i(\boldsymbol{\alpha}^N)}{\partial \alpha_{j,t}}\right)^{-1} - \frac{1}{1-\delta} \\ &= -\frac{\delta\eta}{1-\delta\rho} - \frac{\delta}{1-\delta} < 0 \end{aligned}$$

where the second equality follows from $\alpha^N = \operatorname{argmax} \pi_i(\alpha_i, \boldsymbol{\alpha}^N{}_{-i})$. $f(\underline{\pi}) = 0$ and $f'(\underline{\pi}) < 0$ imply that there exists $\epsilon > 0$ such that $f(\underline{\pi} + \epsilon) < 0$. Thus, $\{\underline{\pi} + \epsilon\}_{t=1}^{\infty}$ satisfy inequalities (8) for $t = 1$ and, consequently, for $t \geq 2$ because $r_t \geq r_1$. Let $\pi' = \underline{\pi} + \epsilon$. Then, $v^*(\underline{\pi}) \geq \pi'/(1-\delta)$. □

Using π' in Lemma 4, I define the interval X and the set of functions $C(X)$ as

$$
\begin{aligned}
X &: [\underline{\pi}, \bar{\pi}] \\
C(X) &: \text{the set of bounded, continuous, and weakly increasing} \\
&\quad \text{functions } f : X \to R \text{ with the sup norm that are weakly concave on } X \\
&\quad s.t. \begin{cases} \pi'/(1-\delta) \leq f(x) \leq \bar{\pi}/(1-\delta) & x \leq \gamma \\ f(x) = \bar{\pi}/(1-\delta) & x \geq \gamma \end{cases}
\end{aligned}
$$

where γ is implicitly defined by the following equation.

$$
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{\pi}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\gamma = \frac{\bar{\pi}}{1-\delta} \tag{9}
$$

Assumption 4 and the definition of Case 1 imply $\underline{\pi} < \gamma < \bar{\pi}$. On $C(X)$, I define the operator T by

$$
Tf(x) = \max_{y \in \Gamma(x;f)} \left[y - \mathbb{1}\{x > y\} \cdot \eta(x-y) + \delta f(\rho x + [1-\rho]y) \right]
$$

where

$$
\begin{aligned}
\Gamma(x;f) = \Bigg\{ x \in [\underline{\pi}, \bar{\pi}] : &\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \mathbb{1}\{x > y\} \cdot \eta \left[\min\{x, b(y)\} - y\right] \\
&+ \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq y + \delta f(\rho x + [1-\rho]y) \Bigg\}
\end{aligned}
$$

Given function f, I obtain properties of Tf. The policy function, $g(x; f)$, is

$$
g(x;f) = \operatorname*{argmax}_{y \in \Gamma(x;f)} \; y - \mathbb{1}\{x > y\} \cdot \eta(x-y) + \delta f(\rho x + [1-\rho]y)
$$

Since f is a weakly increasing function, $g(x; f) = \max \Gamma(x; f)$. By manipulating the condition of

$\Gamma(x; f)$, let $h(x, y; f)$ be

$$h(x, y; f) = \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \mathbb{1}\{x > y\} \cdot \eta \left[\min\{x, b(y)\} - y\right] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x - y - \delta f(\rho x + [1-\rho] y)$$

y belongs to $\Gamma(x; f)$ if and only if $h(x, y; f) \le 0$. Then, I let $\tilde{h}(x; f)$ be

$$\tilde{h}(x; f) = h(x, x; f) = \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(x) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x - x - \delta f(x)$$

$\tilde{h}(x; f)$ is negative at $\underline{\pi}$ as:

$$\begin{aligned}
\tilde{h}(\underline{\pi}; f) &= \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\underline{\pi}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\underline{\pi} - \underline{\pi} - \delta f(\underline{\pi}) \\
&\le \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\underline{\pi} + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\underline{\pi} - \underline{\pi} - \frac{\delta\pi'}{1-\delta} \\
&= -\frac{\delta}{1-\delta}(\pi' - \pi) < 0
\end{aligned}$$

where the inequality follows rom $b(\underline{\pi}) = \underline{\pi}$ and $f(\underline{\pi}) \ge \pi'/(1-\delta)$. Also, $\tilde{h}(x; f)$ is non-positive at $\bar{\pi}$ by construction of Case 1. The strict convexity of $b(\cdot)$ and the weak concavity of $f(\cdot)$ imply $\tilde{h}(x; f)$ is strictly convex in x. These imply that, for all $x \in X$, $\tilde{h}(x, f) \le 0$. In turn, this implies that, for all $x \in X$, $g(x; f) \ge x$. This result eliminates the indicator function because$\mathbb{1}\{x > g(x; f)\} = 0$.

$h(x, g(x); f) = 0$ **on** $[\underline{\pi}, \gamma]$

I now show that the equality of $h(x, y; f) \leq 0$ holds with $y = g(x; f) = \max \Gamma(x; f)$ for all $x \leq \gamma$. From the result above, given $x < \gamma$,$h(x, x; f) = \tilde{h}(x; f) < 0$. On the other hand,

$$
\begin{aligned}
h(x, \bar{\pi}; f) &= \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{\pi}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x - \bar{\pi} - \delta f(\rho x + [1-\rho]\bar{\pi}) \\
&= \frac{\delta\eta\rho}{1-\delta\rho}(\gamma - \bar{\pi}) + \delta f(\rho\gamma + [1-\rho]\bar{\pi}) - \delta f(\rho x + [1-\rho]\bar{\pi}) > 0
\end{aligned}
$$

where the second inequality follows from eq. (9) and f weakly increasing. Since $b(\cdot)$ and $-f(\cdot)$ are convex and continuous functions, given x, $h(x, y; f)$ is convex and continuous in y. Given this convexity, continuity, that $h(x, x; f) < 0$ and that $h(x, \bar{\pi}; f) > 0$, $h(x, g(x; f); f) = 0$.

Tf **and** g **are strictly increasing on** $[\underline{\pi}, \gamma]$

$Tf(x)$ strictly increases in x on $[\underline{\pi}, \gamma]$ because $g(x)$ strictly increases on $[\underline{\pi}, \gamma]$ which follows from $g(x) = \max \Gamma(x; f)$.

Tf **and** g **are strictly concave on** $[\underline{\pi}, \gamma]$

$Tf(x)$ is strictly concave on $(\underline{\pi}, \gamma)$ because $g(x)$ is strictly concave on $(\underline{\pi}, \gamma)$, as below. For $x_1, x_2 \in [\underline{\pi}, \gamma]$ such that $x_1 \neq x_2$, $g(x_1)$ and $g(x_2)$ satisfy the following equations, respectively.

$$
\begin{aligned}
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_1)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x_1 &= g(x_1) + \delta f(\rho x_1 + [1-\rho]g(x_1)) \quad (10) \\
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_2)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x_2 &= g(x_2) + \delta f(\rho x_2 + [1-\rho]g(x_2))
\end{aligned}
$$

Combining these two equations with a weight $\theta \in (0,1)$ yields

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{[1-\theta]b(g(x_1)) + \theta b(g(x_2))\} + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}([1-\theta]x_1 + \theta x_2)$$
$$= (1-\theta)g(x_1) + \theta g(x_2) + (1-\theta)\,\delta f\,(\rho x_1 + [1-\rho]g(x_1)) + \theta\delta f\,(\rho x_2 + [1-\rho]g(x_2))$$

Since $g(x)$ strictly increases on $[\underline{\pi}, \gamma]$, $g(x_1) \neq g(x_2)$. It follows from the convexity of $b(\cdot)$ and the concavity of $f(\cdot)$ that:

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{b([1-\theta]g(x_1) + \theta g(x_2))\} + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}([1-\theta]x_1 + \theta x_2)$$
$$< (1-\theta)g(x_1) + \theta g(x_2) + \delta f\,(\rho\,\{[1-\theta]x_1 + \theta x_2\} + (1-\rho)\,\{[1-\theta]g(x_1) + \theta g(x_2)\}) \quad (11)$$

Since $[1-\theta]x_1 + \theta x_2 \in (\underline{\pi}, \gamma)$, $g([1-\theta]x_1 + \theta x_2) \in (\underline{\pi}, \gamma)$ satisfies the following equality.

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{b(g([1-\theta]x_1 + \theta x_2))\} + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}([1-\theta]x_1 + \theta x_2)$$
$$= g([1-\theta]x_1 + \theta x_2) + \delta f\,(\rho\,\{[1-\theta]x_1 + \theta x_2\} + (1-\rho)g([1-\theta]x_1 + \theta x_2)) \quad (12)$$

Eq. (11) and (12) imply $(1-\theta)g(x_1) + \theta g(x_2) < g([1-\theta]x_1 + \theta x_2)$. That is, $g(x)$ is strictly concave on $(\underline{\pi},\gamma)$. Consequently, $Tf(x)$ is strictly concave on $(\underline{\pi},\gamma)$.

Properties on $[\gamma, \bar{\pi}]$

For the properties on $[\gamma, \bar{\pi}]$, for all $x \geq \gamma$, $h(x, \bar{\pi}; f) \leq 0$ follows from eq. (9), and $\bar{\pi} \in \Gamma(x; f)$. It immediately implies $g(x; f) = \bar{\pi}$ and $Tf(x) = \bar{\pi}/(1-\delta)$. These also imply that $g(x; f)$ and $Tf(x)$ are weakly concave and weakly increasing on X.

Continuity

Continuities of $g(x)$ and $Tf(x)$ on $(\underline{\pi}, \bar{\pi})$ is implied by their weak concavities. Also, they are continuous on $\bar{\pi}$. As for the continuity on $\underline{\pi}$, I prove that of $g(x)$ first. I take a sequence, $\{x_t\} \subset [\underline{\pi}, \gamma]$, that converges to $\underline{\pi}$ as $t \to \infty$. Then, for all x_t,

$$g(x_t) = \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_t)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x_t - \delta f(\rho\underline{\pi} + [1-\rho]g(x_t))$$

It follows from the continuity of $b(\cdot)$ and $f(\cdot)$ that

$$\lim_{x_t \to \underline{\pi}} g(x_t) = \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\lim_{x_t \to \underline{\pi}} g(x_t)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} \\ - \frac{\delta\eta\rho}{1-\delta\rho}\underline{\pi} - \delta f(\rho\underline{\pi} + [1-\rho] \lim_{x_t \to \underline{\pi}} g(x_t))$$

This implies $\lim_{x_t \to \underline{\pi}} g(x_t) = g(\underline{\pi})$.

Pointwise Convergence

Given the results above, $T : C(X) \to C'(X)$ where $C'(x)$ is defined as:

$C'(X)$: the set of bounded, continuous, weakly increasing, and weakly concave functions $f : X \to R$ with the sup norm that are strictly increasing and strictly concave on $[\underline{\pi}, \gamma]$

$$s.t. \begin{cases} \underline{\pi}'/(1-\delta) \le f(x) \le \bar{\pi}/(1-\delta) & x \le \gamma \\ f(x) = \bar{\pi}/(1-\delta) & x \ge \gamma \end{cases}$$

which is a subset of $C(X)$ and. Using this operator T, let

$$\begin{cases} f_1(x) & = \frac{\bar{\pi}}{1-\delta} \quad \text{for all} \quad x \in X \\ f_{n+1} & = T f_n \quad \text{for} \quad n \in \mathbb{N} \end{cases}$$

be a sequence of functions produced by T. f_1 is in $C(X)$. It follows that

$$\begin{cases} f_2(x) < f_1(x) & x < \gamma \\ f_2(x) = f_1(x) & x \geq \gamma \end{cases}$$

For $x < \gamma$, given $f(x) < f'(x)$, $g(x; f) = \max \Gamma(x; f) < \max \Gamma(x; f') = g(x, f')$, and it immediately follows $Tf(x) < Tf'(x)$. By this monotonicity of T, $f_3 = Tf_2 < Tf_1 = f_2$, and, by repeating this operation, $n \in \mathbb{N}$,

$$\begin{cases} f_n(x) < \cdots < f_1(x) & x < \gamma \\ f_n(x) = \cdots = f_1(x) & x \geq \gamma \end{cases}$$

Since $\{f_n\}_{n \in \mathbb{N}}$ is bounded, there exists $f^*(x)$ such that $f_n(x) \to f^*(x)$ as $n \to \infty$.

Uniform Convergence

I show the convergence is uniform. For all $n + 1$, and $y < x \in X$,

$$|f_{n+1}(x) - f_{n+1}(y)| = |Tf_n(x) - Tf_n(y)| = \begin{cases} 0 & x, y \geq \gamma \\ \frac{\bar{\pi}}{1-\delta} - Tf_n(y) & y < \gamma \leq x \\ Tf_n(x) - Tf_n(y) & y < x < \gamma \end{cases}$$

When $y < x < \gamma$,

$$
\begin{aligned}
|f_{n+1}(x) - f_{n+1}(y)| \leq & \Big| \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x; f_n)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \\
& - \left\{\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(y; f_n)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} y\right\} \Big| \\
= & |x - y| \cdot \left| \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x; f_n)) - b(g(y; f_n))}{x - y} - \frac{\delta\eta\rho}{1-\delta\rho} \right| \\
< & |x - y| \cdot \max\left\{ \left| \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b'(\bar{\pi}) \max_{x,y} \frac{g(x) - g(y)}{x - y} - \frac{\delta\eta\rho}{1-\delta\rho} \right|, \right. \\
& \left. \left| \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b'(\underline{\pi}) \min_{x,y} \frac{g(x) - g(y)}{x - y} - \frac{\delta\eta\rho}{1-\delta\rho} \right| \right\}
\end{aligned} \tag{13}
$$

where the last inequality follows from the convexity of $b(\cdot)$. I show $[g(x) - g(y)]/(x - y)$ on the RHS is bounded. We have $h(x_1, g(x_2); f) - h(x_1, g(x_1); f) > 0$ for $x_1 < x_2 < \gamma$. That is,

$$
\begin{aligned}
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_1)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x_1 &= g(x_1) + \delta f(\rho x_1 + [1-\rho] g(x_1)) \\
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_2)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x_1 &> g(x_2) + \delta f(\rho x_1 + [1-\rho] g(x_2))
\end{aligned}
$$

It follows

$$
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x_2)) - b(g(x_1))}{g(x_2) - g(x_1)} - 1 - \delta \frac{f(\rho x_1 + [1-\rho] g(x_2)) - f(\rho x_1 + [1-\rho] g(x_1))}{g(x_2) - g(x_1)} > 0 \tag{14}
$$

Also, it follows from $h(x_2, g(x_2); f) - h(x_1, g(x_1); f) = 0$ that

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x_2)) - b(g(x_1))}{g(x_2) - g(x_1)} \frac{g(x_2) - g(x_1)}{x_2 - x_1} - \frac{\delta\eta\rho}{1-\delta\rho} - \frac{g(x_2) - g(x_1)}{x_2 - x_1} \\
&- \delta \frac{f(\rho x_2 + [1-\rho]g(x_2)) - f(\rho x_1 + [1-\rho]g(x_2))}{x_2 - x_1} \\
&- \delta \frac{f(\rho x_1 + [1-\rho]g(x_2)) - f(\rho x_1 + [1-\rho]g(x_1))}{g(x_2) - g(x_1)} \frac{g(x_2) - g(x_1)}{x_2 - x_1} = 0 \\
\Longleftrightarrow & \left\{ \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x_2)) - b(g(x_1))}{g(x_2) - g(x_1)} - 1 \right. \\
&\left. - \delta \frac{f(\rho x_1 + [1-\rho]g(x_2)) - f(\rho x_1 + [1-\rho]g(x_1))}{g(x_2) - g(x_1)} \right\} \frac{g(x_2) - g(x_1)}{x_2 - x_1} \\
&- \delta \frac{f(\rho x_2 + [1-\rho]g(x_2)) - f(\rho x_1 + [1-\rho]g(x_2))}{x_2 - x_1} = \frac{\delta\eta\rho}{1-\delta\rho}
\end{aligned}$$

The combination of $\rho x_1 + [1-\rho]g(x_2) > \underline{\pi}$ and the concavity of f implies that the second term on the LHS of the last equality is bounded. The RHS is also bounded. Consequently, given inequality (14), $[g(x_2) - g(x_1)]/[x_2 - x_1]$ must be bounded. Consequently, there exists $|f_{n+1}(x) - f_{n+1}(y)| < K|x - y|$ for all $x, y < \gamma$ for some $K < \infty$ because of inequality (13).

When $y < \gamma \leq x$,

$$\begin{aligned}
|f_{n+1}(x) - f_{n+1}(y)| =& \left| (x-y) \frac{\gamma - y}{x - y} (Tf_n(\gamma) - Tf_n(x)) \right| \\
\leq& |x - y| \left| \frac{\gamma - y}{x - y} \right| \\
&\cdot \max \left\{ \left| \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b'(\bar{\pi}) \max_{x,y} \frac{g(x) - g(y)}{x - y} - \frac{\delta\eta\rho}{1-\delta\rho} \right|, \right. \\
&\left. \left| \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b'(\underline{\pi}) \min_{x,y} \frac{g(x) - g(y)}{x - y} - \frac{\delta\eta\rho}{1-\delta\rho} \right| \right\}
\end{aligned}$$

Similarly to the previous case, this inequality implies $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous. Also, $\{f_n\}_{n\in\mathbb{N}}$ is bounded, and X is closed and bounded. It follows that the convergence of $\{f_n\}_{n\in\mathbb{N}}$ is uniform by the Ascoli–Arzelà theorem. Then, since $(C(X), \|\cdot\|_\infty)$ is a compact set, $f^* \in C(X)$.

f^* satisfies the function equation

I prove that f^* satisfies the functional equation by showing that f_n converges to Tf^*. It follows from the uniform convergence that for any $\varepsilon > 0$, there exists N such that $\|f_n - f^*\| < \varepsilon$ for all $n > N$. In addition, $f^* \leq f_n$ and, therefore, $f_n < f^* + \varepsilon$ on X for $n > N$. By the monotonicity, $Tf_n < T(f^* + \varepsilon)$. Then,

$$\begin{aligned}
\|Tf_n - Tf^*\| &= \sup_{x \in X} |Tf_n(x) - Tf^*(x)| \\
&< \sup_{x \in X} |Tf^*(x + \varepsilon) - Tf^*(x)| \quad (n > N) \\
&= \sup_{x \in X}\; g(x, f^* + \varepsilon) + \delta f^*(\rho x + [1 - \rho] g(x, f^* + \varepsilon)) + \delta\varepsilon \\
&\qquad\qquad - \{g(x, f^*) + \delta f^*(\rho x + [1 - \rho] g(x, f^*))\} \\
&= \sup_{x \in X}\; l(x, f^*, \varepsilon)
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
l(x, f^*, \varepsilon) = & g(x, f^* + \varepsilon) + \delta f^*(\rho x + [1 - \rho] g(x, f^* + \varepsilon)) + \delta\varepsilon \\
& - \{g(x, f^*) + \delta f^*(\rho x + [1 - \rho] g(x, f^*))\}
\end{aligned}$$

First, I show that $l(x, f^*, \varepsilon) \leq l(\underline{\pi}, f^*, \varepsilon)$. I let $\Delta g(x) = g(x, f^* + \varepsilon) - g(x, f^*)$ and $\Delta f^*(x) = f^*(\rho x + [1 - \rho] g(x, f^* + \varepsilon)) - f^*(\rho x + [1 - \rho] g(x, f^*))$. Then, $l(x, f^*, \varepsilon)$ increases with $\Delta g(x)$ and $\Delta f^*(x)$.

I show that, given ε, both $\Delta g(x)$ and $\Delta f^*(x)$ can be maximized by $x = \underline{\pi}$. The following two equations hold for $x, x + \epsilon \leq \gamma$.

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x, f^* + \varepsilon)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x$$
$$= g(x, f^* + \varepsilon) + \delta f^*(\rho x + [1-\rho]g(x, f^* + \varepsilon)) + \delta\varepsilon$$
$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x, f^*)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x$$
$$= g(x, f^*) + \delta f^*(\rho x + [1-\rho]g(x, f^*))$$

By combining these two equations,

$$Y(x, \Delta g(x)) \equiv \left[1 - \frac{h\delta\rho}{1-\delta(1-\rho)}\right]\{b(g(x, f^*) + \Delta g(x)) - b\,(g(x, f^*))\}$$
$$- \Delta g(x) - \delta\,\{f^*(\rho x + [1-\rho]g(x, f^*) + \rho\Delta g(x)) - f^*\,(\rho x + [1-\rho]g(x, f^*))\} = \delta\varepsilon$$

I let $\bar{x} = \mathrm{argmax}_{x \in \{x \in X | Y(x, \Delta g(x) = \delta\varepsilon\}} \Delta g(x)$. Suppose $\Delta g(\bar{x}) > \underline{\pi}$. Then, since $b(\cdot)$ is convex, $f^*(\cdot)$ is weakly concave, and $g(x)$ increases in x,

$$\delta\varepsilon = Y(\bar{x}, \Delta g(\bar{x})) > Y(\underline{\pi}, \Delta g(\bar{x}))$$

$Y(\underline{\pi}, \Delta g(x))$ is convex in $\Delta g(x)$, and $Y(\underline{\pi}, 0) = 0$. Because of the convexity, there exits $\Delta g > \Delta g(\bar{x})$ such that $Y(\underline{\pi}, \Delta g) = \delta\epsilon$. This contradicts $\bar{x} = \mathrm{argmax}_{x \in \{x \in X | Y(x, \Delta g(x) = \delta\varepsilon\}} \Delta g(x)$. Thus, $\max \Delta g(x) = \Delta g(\underline{\pi})$. As for $\Delta f^*(x)$, because f^* is weakly concave, $g(x, f^*)$ weakly increases in x, and $\max \Delta g(x) = \Delta g(\underline{\pi})$,

$$\Delta f^*(x) = f^*(\rho x + [1-\rho]g(x, f^*) + \rho\Delta g(x)) - f^*(\rho x + [1-\rho]g(x, f^*))$$
$$\leq f^*(\rho\underline{\pi} + [1-\rho]g(\underline{\pi}, f^*) + \rho\Delta g(\underline{\pi})) - f^*(\rho\underline{\pi} + [1-\rho]g(\underline{\pi}, f^*)) = \Delta f^*(\underline{\pi})$$

These results imply

$$\|Tf_n - Tf^*\| < \max_{x\in X} l(x, f^*, \varepsilon) \le l(\underline{\pi}, f^*, \varepsilon)$$
$$= g(\underline{\pi}, f^* + \varepsilon) + \delta f^*(\rho\underline{\pi} + [1-\rho]g(\underline{\pi}, f^* + \varepsilon)) + \delta\varepsilon$$
$$- \{g(\underline{\pi}, f^*) + \delta f^*(\rho x + [1-\rho]g(\underline{\pi}, f^*))\}$$

Next, I prove that $g(\underline{\pi}, f)$ is continuous in f at $f = f^*$. On $x = \underline{\pi}$, the following condition holds for small ε, where $\tilde{g}(\varepsilon)$ denotes $g(\underline{\pi}, f^* + \varepsilon)$,

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\tilde{g}(\varepsilon)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\underline{\pi}$$
$$= \tilde{g}(\varepsilon) + \delta f^*(\rho\underline{\pi} + [1-\rho]\tilde{g}(\varepsilon)) + \delta\varepsilon$$

This equation can be rewritten as $Z(g'(\varepsilon)) = \varepsilon$ where $Z(x) = (1/\delta)\cdot[1 - \delta\eta(1-\rho)/(1-\delta\rho)]\, b(x) + \{\delta/(1-\delta) + \delta\eta/[1-\delta\rho]\}\underline{\pi} - \delta\eta\rho/(1-\delta\rho)\underline{\pi} - x - \delta f(\rho\underline{\pi} + [1-\rho]x)$. $Z(\cdot)$ is a continuous and convex function and $Z(\tilde{g}(0)) = 0$, which implies $\tilde{g}(\epsilon)$ strictly increases in ε and continuous in ε. Finally, I show Tf_n converges to Tf^*. Substitute (**??**) into (**??**),

$$\|Tf_n - Tf^*\| < l(\underline{\pi}, f^*, \varepsilon) = \tilde{g}(\varepsilon) + \delta f^*(\rho\underline{\pi} + [1-\rho]\tilde{g}(\varepsilon)) + \delta\varepsilon$$
$$- \{\tilde{g}(0) + \delta f^*(\rho\underline{\pi} + [1-\rho]\tilde{g}(0)) + \delta\cdot 0\} \tag{16}$$

Since $\tilde{g}$ and f^* are both continuous, $\tilde{g}(\varepsilon) + \delta f^*([1-\rho]\underline{\pi} + \rho q(\varepsilon)) + \delta\varepsilon$ is continuous in ε. Therefore, for any $\omega > 0$, there exist $\zeta > 0$ such that for all $\varepsilon < \zeta$,

$$\|Tf_n - Tf^*\| < \tilde{g}(\varepsilon) + \delta f^*(\rho\underline{\pi} + [1-\rho]\tilde{g}(\varepsilon)) + \delta\varepsilon$$
$$- \{\tilde{g}(0) + \delta f^*(\rho\underline{\pi} + [1-\rho]\tilde{g}(0)) + \delta\cdot 0\} < \omega \tag{17}$$

Therefore, Tf_n uniformly converges to Tf^*. This immediately implies f_n converges to Tf^*. Since f_n converges also to f^*, $f^* = Tf^*$.

Since f^* is the solution to the functional equation whose value is bounded, $v^* = f^*$. Additionally, since $v^* = f^* \in C(X)$, $v^* = Tv^* \in C'(X)$ and v^* belongs to $C'(X)$. The policy function $g^*(x) = g(x, v^*)$ has the same properties as $g(x, f)$ for $f \in C(X)$.

F Proof of Proposition 3.

I prove Proposition (3) in three steps. First, I characterize the value function and the policy function of a modified problem that does not have the loss-evading deviation incentive or the diminishing future cooperation value. Second, using the results of the first modified problem, I characterize the value function and the policy function of another modified problem that does not have the loss-evading deviation incentive. Finally, using the results of the second modified problem, I characterize the value function and the policy function of problem (4).

First, I work on the following modified problem.

$$w_1^*(r_1) = \max_{\{\pi_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1}\pi_t \tag{18}$$

$$s.t.\ \pi_t \in \Omega_1(r_t, w_1^*) \text{ for all } t$$

$$r_{t+1} = (1-\rho)r_t + \rho\pi_t$$

where

$$\Omega_1(r_t, w^*) = \left\{\pi_t \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \leq w_1^*(r_t)\right\}$$

This problem (18) does not have the loss utility terms with the indicator functions: $\mathbb{1}\{r_t > \pi_t\} \cdot \eta(r_t - \pi_t)$ in the objective function and $\mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta\left[r_t - b(\pi_t)\right]$ in the constraint of the feasible set. Problem (18) satisfies the corresponding functional equation

$$w_1^*(r) = \max_{y \in \Omega_1(r, w_1^*)} [y + \delta w_1^*(\rho r + [1-\rho]y)]$$

and it can be solved in the same way as the $[\underline{\pi}, \gamma]$ part of the proof of Proposition (2). Let

$$\begin{aligned} X :& [\underline{\pi}, \bar{\pi}] \\ C(X) :& \text{the set of bounded, continuous, and weakly increasing} \\ & \text{functions } f : X \to R \text{ with the sup norm that are weakly concave on } X \\ & s.t.\ \forall x \in X,\ \underline{\pi}\text{'}/(1-\delta) \le f(x) \le \bar{\pi}/(1-\delta) \end{aligned}$$

On $C(X)$, I define the operator T by

$$Tf(x) = \max_{y \in \Omega_1(x; f)} [y + \delta f(\rho x + [1-\rho]y)]$$

where

$$\Omega_1(x; f) = \left\{ x \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \le y + \delta f(\rho x + [1-\rho]y) \right\}$$

Given function f, let the policy function, $g_{w1} : X \to Y$ be:

$$g_{w1}(x; f) = \underset{y \in \Omega_1(x; f)}{\text{argmax}}\ y + \delta f(\rho x + [1-\rho]y)$$

Since f is a weakly increasing function, $g_{w1}(x; f) = \max \Omega_1(x; f)$. I obtain the following properties following the same steps as Proposition (2),

1. $w_1^*(r)$ and $g_{w1}^*(r)$ are bounded and continuous on X.

2. $w_1^*(r)$ and $g_{w1}^*(r)$ are strictly increasing and strictly concave on X.

3. $r < g_{w1}^*(r) < \bar{z}$ on $[\underline{\pi}, \bar{z})$, $g_{w1}^*(\bar{z}) = \bar{z}$, and $\bar{z} < g_{w1}^*(r) < r$ on $(\underline{\pi}, \bar{z}]$ where $\bar{z}$ is implicitly defined as

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{z}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}\bar{z} = \bar{z} + \delta\frac{\bar{z}}{1-\delta} \tag{19}$$

4. $w_1^*(r) > r/(1-\delta)$ on $[\underline{\pi}, \bar{z})$, $w_1^*(\bar{z}) = \bar{z}/(1-\delta)$, and $w_1^*(r) < r/(1-\delta)$ on $(\underline{\pi}, \bar{z}]$

Second, I analyze another modified problem.

$$w_2^*(r_1) = \max_{\{\pi_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \left[\pi_t - \mathbb{1}\{r_t > \pi_t\} \cdot \eta(r_t - \pi_t)\right] \tag{20}$$

$$s.t.\ \pi_t \in \Omega_2(r_t, w_2^*) \text{ for all } t$$

$$r_{t+1} = (1-\rho)r_t + \rho\pi_t$$

where

$$\Omega_2(r_t, w_2^*) = \left\{\pi_t \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\pi_t) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r_t \leq \pi_t + \delta w_2^*(r_{t+1})\right\}$$

This problem has loss utilities in the objective function, and, consequently, the present value of future cooperation $\delta w_2^*(r_{t+1})$ on the RHS of the constraint in Ω_2 counts future loss utilities. However, the short-term gain of a deviation on the LHS of the constraint does not include that of avoiding a loss utility in the current period, $\mathbb{1}\{r_t > \pi_t\} \cdot \eta\left[\min\{r_t, b(\pi_t)\} - \pi_{v,t}\right]$. I obtain Lemma 5.

Lemma 5. For all $r \in [\underline{\pi}, \bar{z}]$, $w_2^*(r) = w_1^*(r)$ and $g_{w2}^*(r) = g_{w2}^*(r)$ for all $r \in [\underline{\pi}, \bar{z}]$. For all $r \in [\underline{\pi}, \bar{z}]$, $w_2^*(r) = \bar{z}/(1-\delta) - \eta(x - \bar{z})/(1-\delta\rho)$ and $g_{w2}^*(r) = \bar{z}$.

Proof. The function equation of problem (20) is

$$w_2^*(r) = \max_{y \in \Omega_2(r, w_2^*)} \left[y - \mathbb{1}\{r > y\} \cdot \eta(r-y) + \delta w_2^*(\rho r + [1-\rho]y)\right]$$

I verify that $w_2^*(r)$ of Lemma 5 satisfies this function equation. Suppose that for all $r \leq \bar{z}$, $w_2^*(r) = w_1^*(r)$ and, for all $r > \bar{z}$, $w_2^*(r) = \bar{z}/(1-\delta) - \eta(x-\bar{z})/(1-\delta\rho)$. Then, the RHS of the function equation increases in y for all $r \in [\underline{\pi}, \bar{\pi}]$, and, therefore, $g_{w2}^*(r) = \max \Omega_2(r, w_2^*)$. Notice $g_{w1}^*(r) \in \Omega_2(r, w_2^*)$ for $r \leq \bar{z}$. Also, for all $r \in [\underline{\pi}, \bar{\pi}]$, $\Omega_2(r, w_2^*) \subset \Omega_1(r, w_1^*)$ because $w_2^*(r) < \bar{z}/(1-\delta) < w_1^*(r)$ for all $r > \bar{z}$. Thus, $g_{w2}^*(r) = \max \Omega_2(r, w_2^*) = g_{w1}^*(r)$. For $r > \bar{z}$, the constraint of Ω_2 becomes

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r \leq y + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(\rho r + [1-\rho]y - \bar{z})\right] \\
\iff &\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} \bar{z} \leq y + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta(1-\rho)}{1-\delta\rho}(y - \bar{z})\right]
\end{aligned} \tag{21}$$

Equality (19) and inequality (21) imply $\bar{z} \in \Omega_2(r, w_2^*)$. Suppose there exists $y > \bar{z}$ such that $y \in \Omega(r, w_2^*)$. Then, it follows from inequality (21) that

$$\begin{aligned}
\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} y &\leq y + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(y - \bar{z})\right] \\
&\leq y + \delta w_1(y)
\end{aligned}$$

The last inequality implies $y \in \Omega_1(r, w_1^*)$, which contradicts $g_{w1}^*(r) = \max \Omega_1(w_1^*) < r$ for all $r > \bar{z}$. Thus, given $r > \bar{z}$, $y \notin \Omega_2(r, w_2^*)$ for all $y > \bar{z}$; therefore, $g_{w2}^*(r) = \max \Omega_2(r, w_2^*) = \bar{z}$. These results of $g_{w2}^*(r)$ verify that for all $r \leq \bar{z}$, $w_2^*(r) = g_{w2}^*(r) + \delta w_2^*(\rho r + (1-\rho) g_{w2}^*(r)) = w_1^*(r)$ and, for all $r > \bar{z}$, $w_2^*(r) = g_{w2}^*(r) - \eta(r - \bar{z}) + \delta w_2^*(\rho r + (1-\rho) g_{w2}^*(r)) = \bar{z}/(1-\delta) - \eta(x-\bar{z})/(1-\delta\rho)$. □

Finally, I compare v^* to w_2^*. The only difference between problem (3) and the modified problem (20) is that the constraint of the second modified problem 20 does not have $\mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta\,[r_t - b(\pi_t)] \geq 0$ as a short-term gain. Suppose that for some $r \in [\underline{\pi}, \bar{\pi}]$ $v^*(r) > w_2^*(r)$ and let $\{\pi_t^o\}_{t=1}^{\infty}$ be the optimized path with $r_1 = r$ ($v^*(r_1) = \sum_{t=1}^{\infty} \delta^{t-1}\,[\pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o)]$.) This $\{\pi_t^o\}_{t=1}^{\infty}$ is feasible in the second modified problem 20 with $w_2^*(r_2) = \sum_{t=2}^{\infty} \delta^{t-1}\,[\pi_t^o - \mathbb{1}\{r_t > \pi_t^o\} \cdot \eta(r_t - \pi_t^o)]$ in the constraint, which implies $w_2^*(r) \geq v^*(r)$. This result contradicts $v^*(r) > w_2^*(r)$. Thus,

$v^*(r) \leq w_2^*(r)$ for all $r \in [\underline{\pi},\bar{\pi}]$. It immediately follows that, for $r \leq \bar{z}$, $g^*(r) = g_{w2}^*(r)$, which makes the equality of $v^*(r) \leq w_2^*(r)$ hold and $\mathbb{1}\{r_t > b(\pi_t)\} \cdot \eta\,[r_t - b(\pi_t)]$ equal 0 for all r_t with $r_1 > \bar{z}$. For $r \leq \bar{z}$, suppose $g^*(r) \geq \bar{z}$. It follows that

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g^*(r)) - \mathbb{1}\{r > b(g^*(r))\} \cdot \eta\,[r - b(g^*(r))] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r \leq v^*(r)$$
$$\Longrightarrow \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g^*(r)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} r \leq w_2(r)$$

This contradicts that $\forall r > \bar{z}$, $\forall y > \bar{z}$, $y \notin \Omega_2(r, w_2^*)$. Thus, $g^*(r) < \bar{z}$.

G Proof of Proposition 4

Let

$$\begin{aligned}
X :& [\underline{\pi}, \bar{\pi}] \\
C(X) :& \text{the set of bounded and continuous functions } f : X \to R \\
& \text{with the sup norm that are weakly concave on } [\bar{z}, \bar{\pi}] \\
& s.t.\ \forall x \in [\underline{\pi}, \bar{z}],\ f(x) = w_2^*(x) \\
& \qquad \forall x \in [\bar{z}, \bar{\pi}],\ f(x) \leq \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}) \\
& \qquad \forall x, \forall y \in [\bar{z}, \bar{\pi}]\ s.t.\ x < y,\ \frac{f(y) - f(x)}{y - x} \geq -\frac{1+\eta}{\delta(1-\rho)}
\end{aligned}$$

where w_2^* is that of Appendix F. On $C(X)$, I define the operator T by

$$Tf(x) = \max_{y \in \Gamma(x;f)} \left[y - \mathbb{1}\{x > y\} \cdot \eta(x - y) + \delta f(\rho x + [1-\rho]y)\right]$$

where

$$\Gamma(x; f) = \Big\{x \in [\underline{\pi}, \bar{\pi}] : \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \mathbb{1}\{x > y\} \cdot \eta \left[\min\{x, b(y)\} - y\right] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq y + \delta f(\rho x + [1-\rho]y)\Big\}$$

Given function f, let the policy function, $g : X \to Y$ be:

$$g(x; f) = \operatorname*{argmax}_{y \in \Gamma(x;f)} \; y - \mathbb{1}\{x > y\} \cdot \eta(x - y) + \delta f(\rho x + [1-\rho]y)$$

I denote $g(x; f)$ by $g(x)$ unless it is confusing. When $\bar{z}$ is sufficiently close to $\bar{\pi}$

$g(x; f) = \max \Gamma(x; f)$

Suppose $g(x; f) < \max \Gamma(x; f) = \gamma(x, f)$ for some $x \in (\bar{z}, \bar{\pi}]$. Then,

$$\begin{aligned} 0 &< \{g(x) - \eta(x - g(x)) + \delta f(\rho x + [1-\rho]g(x))\} - \{\gamma(x) - \eta(x - \gamma(x)) + \delta f(\rho x + [1-\rho]\gamma(x))\} \\ &< (1+\eta)\left[g(x) - \gamma(x)\right] + \delta(1-\rho)\left[g(x) - \gamma(x)\right]\left[-\frac{1+\eta}{\delta(1-\rho)}\right] = 0 \end{aligned}$$

This is a contradiction. Thus, $g(x; f) = \max \Gamma(x; f)$.

$f = Tf = w^*(x)$ **for all** $x \leq \bar{z}$

$w_1^*(x) > \bar{z}/(1-\delta) > f(x)$ on $(\bar{z}, \bar{\pi}]$ implies $\Gamma(x, f) \subset \Omega(x, f)$. Also, $g_{w1}^*(x) = \max \Omega_1(x, f) \in \Gamma(x, f)$. Thus, $g^*(x, f) = g_w^*(x)$ on $[\underline{\pi}, \bar{z}]$.

$g(x; f) < \bar{z}$ **for** $x > \bar{z}$

Suppose y$\in \Gamma(x, f)$ for some $x > \bar{z}$ and $y > \bar{z}$. Then,

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \mathbb{1}\{x > y\} \cdot \eta \left[\min\{x, b(y)\} - y\right] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq y + \delta f(\rho x + [1-\rho] y)$$

$$\implies \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq y + \delta w_2^*(\rho x + [1-\rho] y)$$

where I use $f(r) = w_2^*(r)$ for $r \leq \bar{z}$ and $f(r) \leq w_2(r)$ for $r > \bar{z}$. This inequality contradicts that given $r > \bar{z}$, $y \notin \Omega_2(r, w_2^*)$ for all $y > \bar{z}$, which I obtain in Appendix (F). Thus, $\forall x \in (\bar{z}, \bar{\pi}]$, $\forall y \in (\bar{z}, \bar{\pi}]$ $\bar{y} \notin \Gamma(x; f)$. Next, suppose $\bar{z} \in \Gamma(x, f)$ for some $x > \bar{z}$.

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{z}) + \mathbb{1}\{x > \bar{z}\} \cdot \eta \left[\min\{x, b(\bar{z})\} - \bar{z}\right] + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x \leq \bar{z} + \delta f(\rho x + [1-\rho] \bar{z})$$

$$\implies \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(\bar{z}) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} \bar{z} < \bar{z} + \delta \frac{\bar{z}}{1-\delta}$$

This contradicts (19). Thus, $\forall x \in (\bar{z}, \bar{\pi}]$, $\bar{z} \notin \Gamma(x; f)$. Given these results, $g(x; f) < \bar{z}$ for $x > \bar{z}$.

Continuities

The continuity of f, y$\notin \Gamma(x, f)$ for all $y \geq \bar{z}$ for given $x > \bar{z}$, and $g(x) = \max \Gamma(x, f)$, imply that $g(x)$ satisfies

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(y) + \eta(\min\{x, b(y)\} - y) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right] \underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho} x = y + \delta f(\rho x + (1-\rho) y) \tag{22}$$

The continuities of Tf and g follow in the same way as Case 1.

Given Assumption (5), $b(g(x)) > x$

$b(g(x)) - x$ is continuous and $b(g(\bar{z})) - \bar{z} = b(\bar{z}) - \bar{z} > 0$. Then, if $\bar{\pi}$ is sufficiently close to $\bar{z}$, $b(g(x)) > x$ for all $x \in [\bar{z}, \bar{\pi}]$.

The upper limit $Tf(x) < \bar{z}/(1-\delta) - \eta(x-\bar{z})/(1-\delta\rho)$

Then, on $x \in [\bar{z}, \bar{\pi}]$

$$\begin{aligned} Tf(x) &< \bar{z} - \eta(x-\bar{z}) + \delta f(\rho x + (1-\rho)\bar{z}) \\ &\leq \bar{z} - \eta(x-\bar{z}) + \delta\left[\frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(\rho x + (1-\rho)\bar{z} - \bar{z})\right] \\ &= \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x-\bar{z}) \end{aligned}$$

where the first inequality follows from that$\forall x, \forall y \in [\bar{z}, \bar{\pi}]$ $s.t.$ $x < y$, $[f(y) - f(x)]/(y-x) \geq -\eta/[\delta(1-\rho)]$ and $\forall x, \forall y \in [\underline{\pi}, \bar{z}]$ $s.t.$ $x < y$, $[f(y) - f(x)]/(y-x) > 0$

Concavity

The concavity can be obtained in the same way as Case 1.

$$\begin{aligned} &\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{[1-\theta]b(g(x_1)) + \theta b(g(x_2))\} + \eta(([1-\theta]x_1 + \theta x_2) - (1-\theta)g(x_1) - \theta g(x_2)) \\ &\quad + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}([1-\theta]x_1 + \theta x_2) \\ &= (1-\theta)g(x_1) + \theta g(x_2) + (1-\theta)\,\delta f(\rho x_1 + [1-\rho]g(x_1)) + \theta\delta f(\rho x_2 + [1-\rho]g(x_2)) \qquad (23) \end{aligned}$$

where $\theta \in (0,1)$. Given that f is increasing and concave on $[\underline{\pi}, \bar{z}]$ and that f is decreasing and concave on $[\bar{z}, \bar{\pi}]$, f is weakly concave on $[\underline{\pi}, \bar{\pi}]$. It follows from eq.(23), the convexity of $b(\cdot)$, and the weak concavity of $f(\cdot)$ that:

$$\left[1-\frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{b([1-\theta]g(x_1)+\theta g(x_2))\}+\eta\left(\tilde{x}-[(1-\theta)g(x_1)+\theta g(x_2)]\right)$$
$$+\left[\frac{\delta}{1-\delta}+\frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi}-\frac{\delta\eta\rho}{1-\delta\rho}\tilde{x} \tag{24}$$
$$\leq (1-\theta)g(x_1)+\theta g(x_2)+\delta f\left(\rho\tilde{x}+(1-\rho)\{[1-\theta]g(x_1)+\theta g(x_2)\}\right) \tag{25}$$

where $\tilde{x}=[1-\theta]x_1+\theta x_2$. It follows from eq. (22) that:

$$\left[1-\frac{\delta\eta(1-\rho)}{1-\delta\rho}\right]\{b(g(\tilde{x}))\}+\eta(\tilde{x}-g(\tilde{x}))+\left[\frac{\delta}{1-\delta}+\frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi}-\frac{\delta\eta\rho}{1-\delta\rho}\tilde{x}$$
$$=g(\tilde{x})+\delta f\left(\rho\tilde{x}+(1-\rho)g(\tilde{x})\right) \tag{26}$$

Since $g(\tilde{x})=\max\Gamma(x,f)$, eq. (25) and (26) imply

$$(1-\theta)g(x_1)+\theta g(x_2)\leq g(\tilde{x})=g([1-\theta]x_1+\theta x_2)$$

Thus, $g(x)$ is weakly concave. Consequently, so is $Tf(x)$. The concavity, $g(\bar{z})=\bar{z}$, and $g(x)<\bar{z}$ jointly imply $g(x)$ is strictly decreasing on $[\bar{z},\bar{\pi}]$. In turn, this implies $g(x_1)\neq g(x_2)$ for $x_1\neq x_2$ and repeating the same procedure above yields the strict concavity of $g(x)$. Given that f is increasing and strictly concave on $[\underline{\pi},\bar{z}]$ and decreasing and strictly concave on $[\underline{\pi},\bar{\pi}]$, f is strictly concave on $[\underline{\pi},\bar{\pi}]$. Also, $g(x)$ decreasing implies $Tf(x)$ decreasing because of $[f(y)-f(x)]/(y-x)\geq -(1+\eta)/\delta(1-\rho)$.

The slope of Tf

Given $x_1, x_2 > \bar{z}$ such that $x_2 < x_1$, $g(x_2) > g(x_1)$. It follows from eq. (22) for x_1 and from that $g(x_2) \notin \Gamma(x_1, f)$

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_1)) + \eta(x_1 - g(x_1)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x_1$$
$$= g(x_1) + \delta f(\rho x_1 + [1-\rho]g(x_1))$$
$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] b(g(x_2)) + \eta(x_1 - g(x_2)) + \left[\frac{\delta}{1-\delta} + \frac{\delta\eta}{1-\delta\rho}\right]\underline{\pi} - \frac{\delta\eta\rho}{1-\delta\rho}x_1$$
$$> g(x_2) + \delta f(\rho x_1 + [1-\rho]g(x_2))$$

By manipulating these, the following inequality can be obtained.

$$\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x_2)) - b(g(x_1))}{g(x_2) - g(x_1)} - 1 - \eta - \delta\frac{f(\rho x + [1-\rho]g(x_2)) - f(\rho x + [1-\rho]g(x_1))}{g(x_2) - g(x_1)} > 0 \tag{27}$$

It also follows from eq. (22) for x_1 and x_2 that

$$\begin{aligned}
&\left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \cdot \frac{b(g(x_1)) - b(g(x_2))}{g(x_1) - g(x_2)} \cdot \frac{g(x_1) - g(x_2)}{x_1 - x_2} - \frac{\delta\eta\rho}{1-\delta\rho} \\
&= (1+\eta)\frac{g(x_1) - g(x_2)}{x_1 - x_2} - \eta \\
&+ \delta(1-\rho)\frac{f(\rho x_1 + (1-\rho)g(x_1)) - f(\rho x_1 + (1-\rho)g(x_2))}{(1-\rho)g(x_1) - (1-\rho)g(x_2)} \frac{g(x_1) - g(x_2)}{x_1 - x_2} \\
&+ \delta\rho\frac{f(\rho x_1 + (1-\rho)g(x_2)) - f(\rho x_2 + (1-\rho)g(x_2))}{\rho x_1 - \rho x_2}
\end{aligned}$$

This becomes:

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} = \frac{-\eta(1-2\delta\rho)/(1-\delta\rho) + \delta\rho F_1}{[1 - \delta\eta(1-\rho)/(1-\delta\rho)]\,[b(g(x_1)) - b(g(x_2))]\,/\,[g(x_1) - g(x_2)] - 1 - \eta - \delta(1-\rho)F_2} \tag{28}$$

where

$$F_1 = \frac{f(\rho x_1 + (1-\rho)g(x_2)) - f(\rho x_2 + (1-\rho)g(x_2))}{\rho x_1 - \rho x_2}$$
$$F_2 = \frac{f(\rho x_1 + (1-\rho)g(x_1)) - f(\rho x_1 + (1-\rho)g(x_2))}{(1-\rho)g(x_1) - (1-\rho)g(x_2)}$$

We know that the left-side hand of eq. (28) is negative and that the denominator of the right-hand side is positive from inequality (27). This implies that $\delta\rho > 1/2$ is a sufficient condition for $\rho x + (1-\rho)g(x)$ to be greater than $\bar{z}$. Otherwise, $F_1 > 0$ and the right-hand side of eq. (28) become positive. Since $F_2 \geq \min\{0, -(1+\eta)/\delta(1-\rho)\} = -(1+\eta)/\delta(1-\rho)$, the numerator is negative, and the denominator is positive,

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} \leq \frac{-\eta(1-2\delta\rho)/(1-\delta\rho) + \delta\rho F_1}{[1 - \delta\eta(1-\rho)/(1-\delta\rho)]} \frac{g(x_1) - g(x_2)}{b(g(x_1)) - b(g(x_2))} \tag{29}$$

Finally,

$$\begin{aligned}
Tf(x_1) - Tf(x_2) =& g(x_1) - \eta\left[x_1 - g(x_1)\right] + \delta f(\rho x_1 + [1-\rho]g(x_1)) \\
& - \{g(x_2) - \eta\left[x_2 - g(x_2)\right] + \delta f(\rho x_2 + [1-\rho]g(x_2))\} \\
=& \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x_1)) - b(g(x_2))}{g(x_1) - g(x_2)} \frac{g(x_1) - g(x_2)}{x_1 - x_2}(x_1 - x_2) - \frac{\delta\eta\rho}{1-\delta\rho}(x_1 - x_2) \\
\geq& \left[1 - \frac{\delta\eta(1-\rho)}{1-\delta\rho}\right] \frac{b(g(x_1)) - b(g(x_2))}{g(x_1) - g(x_2)} \left[\frac{-\eta(1-2\delta\rho)/(1-\delta\rho) + \delta\rho F_1}{[1 - \delta\eta(1-\rho)/(1-\delta\rho)]}\right] \\
& \cdot \frac{g(x_1) - g(x_2)}{b(g(x_1)) - b(g(x_2))}(x_1 - x_2) - \frac{\delta\eta\rho}{1-\delta\rho}(x_1 - x_2) \\
=& \left[-\eta + \delta\rho F_1\right](x_1 - x_2)
\end{aligned}$$

where the second equality follows from eq. (22) and the first inequality follows from eq. (29) and $b(\cdot)$ being increasing. If $\rho x_2 + (1-\rho)g(x_2) \leq \bar{z}$,

$$-\eta + \delta\rho F_1 > -\eta > -\frac{1+\eta}{\delta(1-\rho)}$$

If $\rho x_2 + (1-\rho)g(x_2) \leq \bar{z}$,

$$-\eta + \delta\rho F_1 > -\eta - \delta\frac{\rho(1+\eta)}{\delta(1-\rho)} = -\frac{\rho+\eta}{1-\rho} > -\frac{1+\eta}{\delta(1-\rho)}$$

In either case, $[Tf(x_1) - Tf(x_2)]/(x_1 - x_2) > -(1+\eta)/\delta(1-\rho)$.

Convergence

Given the results above, $T : C(X) \rightarrow C'(X)$ where $C'(x)$ is defined as:

$$\begin{aligned}
X :&[\underline{\pi}, \bar{\pi}] \\
C'(X) :&\text{the set of bounded, continuous, and strictly concave functions } f : X \rightarrow R \\
&\text{with the sup norm that are strickly decreasing on } [\bar{z}, \bar{\pi}] \\
&s.t.\ \forall x \in [\underline{\pi}, \bar{z}],\ f(x) = w^*(x) \\
&\qquad \forall x \in [\bar{z}, \bar{\pi}],\ f(x) < \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}) \\
&\qquad \forall x, \forall y \in [\bar{z}, \bar{\pi}]\ s.t.\ x < y,\ \frac{f(y) - f(x)}{y - x} > -\frac{1+\eta}{\delta(1-\rho)}
\end{aligned}$$

which is a subset of $C(X)$ and. Using this operator T, let

$$\begin{cases} f_1(x) &= \begin{cases} w^*(x) & x \in [\underline{\pi}, \bar{z}] \\ \frac{\bar{z}}{1-\delta} - \frac{\eta}{1-\delta\rho}(x - \bar{z}) & x \in [\bar{z}, \bar{\pi}] \end{cases} \\ f_{n+1} &= Tf_n \quad \text{for} \quad n \in \mathbb{N} \end{cases}$$

be a sequence of functions produced by T. f_1 is in $C(X)$. It follows that

$$\begin{cases} f_2(x) = f_1(x) = w^*(x) & x \in [\underline{\pi}, \bar{z}] \\ f_2(x) < f_1(x) & x \in (\bar{z}, \bar{\pi}] \end{cases}$$

The rest of the proof follows that of Case 1 becausef_n is strictly concave on $[\underline{\pi}, \bar{\pi}]$.

H Transitory Shocks and Fluctuations of Cooperation

An exogenous transitory payoff shock can shake a high level of cooperation. Consider a group who are enjoying high cooperation at $\bar{z}$, and suppose, after they make the same cooperation decisions in period t ($\pi_t = g(\bar{z}) = \bar{z}$), there is an exogenous one-off payoff shock ϵ such that $\pi_t = \bar{z} + \epsilon$.[12] While this does not affect $\bar{z}$ in the following periods, the reference point in period $t + 1$ is affected. If the shock is negative ($\epsilon < 0$), the reference point in period $t + 1$ becomes lower than $\bar{z}$. This lower reference point lessens the aversion to losing cooperation. In other words, the experience of a bad day makes them tolerant of potential losses in the deviation path. Consequently, the group has to decrease the cooperation level in period $t + 1$, and they gradually raise it back to $\bar{z}$ over periods.

On the other hand, a positive shock hinders cooperation tomorrow; it makes the reference point in period $t + 1$ higher than $\bar{z}$. Suppose the discount factor and the persistence of a reference point are sufficiently low, like in Example 2. The shock can cause a significant drop in cooperation in

[12] I assume ϵ is not too large such that $\underline{\pi} \leq \bar{z} + \epsilon \leq \bar{\pi}$.

period $t + 1$ because of the steep downward slope of g^* in Figure 3. A group that experiences a good time by luck forms a reference point that is not sustainable, and members want to maintain it even for a short term because of loss aversion. This deviation incentive makes it impossible for them to keep even the original level of cooperation, and they suffer low cooperation. Ironically, a windfall today is not beneficial for them tomorrow.

Cooperation with a long history can be more vulnerable to a positive transitory shock. I relax the assumption that the players are at $\bar{z}$ and, instead, suppose that a group starts cooperating with an initial reference point lower than $\bar{z}$ and faces a positive transitory shock in an early period. Then, the reference point r_t is not high, and the reference point in the next period r_{t+1} can be lower than $\bar{z}$. This reference point implies that the cooperation level in period $t + 1$ is higher than that without the shock $g(\rho r_t + (1 - \rho)\left[g(r_{t+1}) + \epsilon\right]) > g(\rho r_t + (1 - \rho)g(r_{t+1}))$. That is, this windfall is beneficial for them not only today but also tomorrow. On the other hand, suppose the shock occurs when the group has repeated cooperation and almost reaches $\bar{z}$. Then, the reference point can exceed $\bar{z}$, and the cooperation can be lower than that without the shock $g(\rho r_t + (1 - \rho)\left[g(r_{t+1}) + \epsilon\right]) < g(\rho r_t + (1 - \rho)g(r_{t+1}))$.